General Mathematics

Learner's Material

This learning resource was collaboratively developed and reviewed by educators from public and private schools, colleges, and/or universities. We encourage teachers and other education stakeholders to email their feedback, comments and recommendations to the Department of Education at action@deped.gov.ph.

We value your feedback and recommendations.

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Learner's Material
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Lesson 1: Functions

Learning Outcome(s): At the end of the lesson, the learner is able to represent real-life situations using functions, including piecewise functions.

Lesson Outline:

1. Functions and Relations
2. Vertical Line Test
3. Representing real-life situations using functions, including piecewise functions.

Definition: A **relation** is a rule that relates values from a set of values (called the **domain**) to a second set of values (called the **range**).

A relation is a set of ordered pairs *(x,y)*.

Definition: A **function** is a relation where each element in the domain is related to only one value in the range by some rule.

A function is a set of ordered pairs *(x,y)* such that no two ordered pairs have the same x-value but different y-values. Using functional notation, we can write f(x) = y, read as "*f* of x is equal to y." In particular, if (1, 2) is an ordered pair associated with the function f, then we say that f(2) = 1.

Example 1. Which of the following relations are functions?

$$f = \{(1,2),(2,3),(3,5),(4,7)\}$$

$$g = \{(1,3),(1,4),(2,5),(2,6),(3,7)\}$$

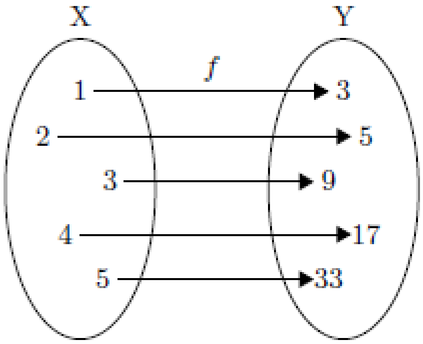
$$h = \{(1,3),(2,6),(3,9),\cdots,(\mathrm{n},3\mathrm{n}),\cdots\}$$

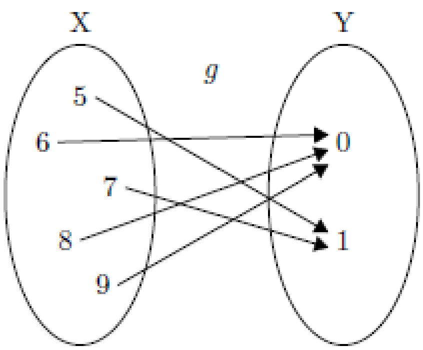
Solution.

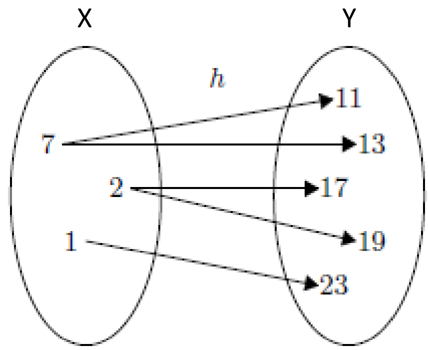
The relations *f* and *h* are functions because no two ordered pairs have the same x-value but different *y*-values. Meanwhile, g is not a function because (1,3) and (1,4) are ordered pairs with the same *x*-value but different y-values.

Relations and functions can be represented by mapping diagrams where the elements of the domain are mapped to the elements of the range using arrows. In this case, the relation or function is represented by the set of all the connections represented by the arrows.

Example 2. Which of the following mapping diagrams represent functions?







Solution.

The relations *f* and *g* are functions because each value *y* in *Y* is unique for a specific value of *x*. The relation *h* is not a function because there is at least one element in *X* for which there is more than one corresponding *y*-value. For example, *x*=7 corresponds to *y* = 11 or 13. Similarly, *x*=2 corresponds to both *y*=17 or 19.

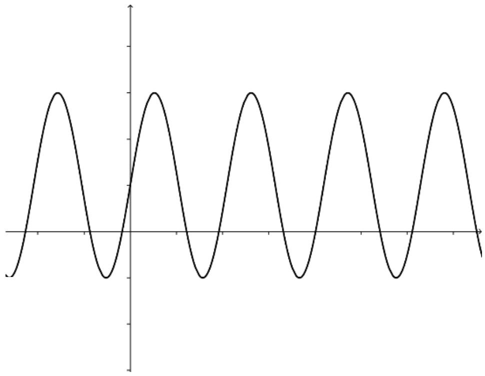
A relation between two sets of numbers can be illustrated by a graph in the Cartesian plane, and that a function passes the vertical line test.

The Vertical Line Test

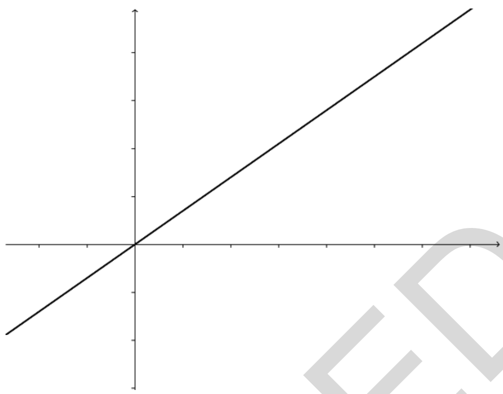
A graph represents a function if and only if each vertical line intersects the graph at most once.

Example 3. Which of the following can be graphs of functions?

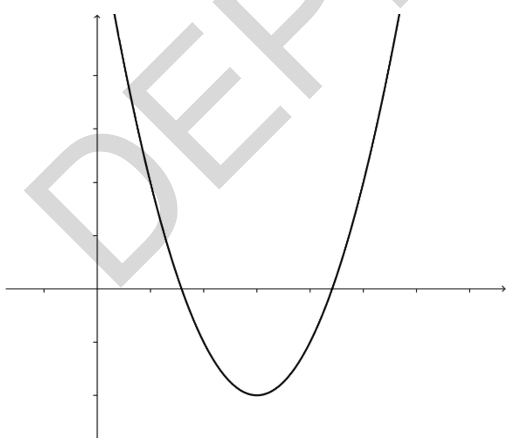
a.)



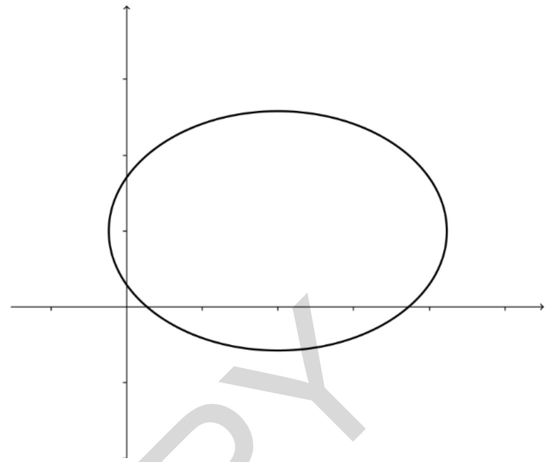
b.)



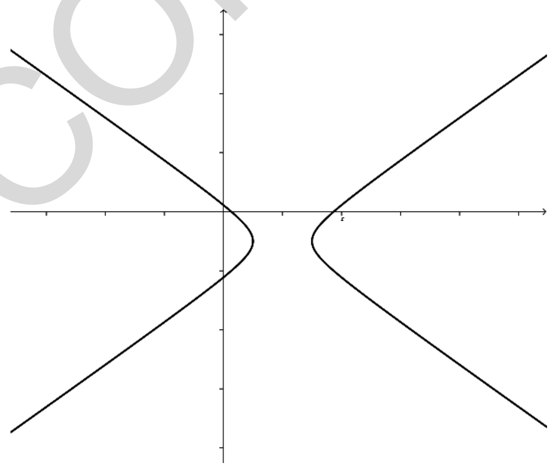
c.)



d.)



e.)



Solution.

Graphs a.), b.), c.) are graphs of functions while d.) and e.) are not because they do not pass the vertical line test.

Important Concepts.

- **Relations** are rules that relate two values, one from a set of inputs and the second from the set of outputs.
- **Functions** are rules that relate only one value from the set of outputs to a value from the set of inputs.

Definition: The **domain** of a relation is the set of all possible values that the variable *x* can take.

Example 4. Identify the domain for each relation using set builder notation.

(a) $y = 2x + 1$
(b) $y = x^2 - 2x + 2$
(c) $x^2 + y^2 = 1$
(d) $y = \sqrt{x+1}$
(e) $y = \frac{2x+1}{x-1}$
(f) $y = \lfloor x \rfloor + 1$ where $\lfloor x \rfloor$ is the greatest integer function.

Solution. The domains for the relations are as follows:

(a) $\{x : x \in \mathbb{R}\}$
(b) $\{x : x \in \mathbb{R}\}$
(c) $\{x : x \in \mathbb{R}, -1 \leq x \leq 1\}$
(d) $\{x : x \in \mathbb{R}, x \geq -1\}$
(e) $\{x : x \in \mathbb{R}, x \neq 1\}$
(f) $\{x : x \in \mathbb{R}\}$

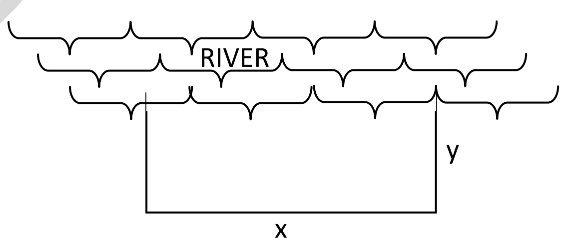
Functions as representations of real-life situations.

Functions can often be used to model real situations. Identifying an appropriate functional model will lead to a better understanding of various phenomena.

Example 5. Give a function C that can represent the cost of buying x meals, if one meal costs P40.

Solution. Since each meal costs P40, then the cost function is C(x) = 40x.

Example 6. One hundred meters of fencing is available to enclose a rectangular area next to a river (see figure). Give a function A that can represent the area that can be enclosed, in terms of x.



Solution. The area of the rectangular enclosure is A = xy. We will write this as a function of x. Since only 100 m of fencing is available, then x + 2y = 100 or y = (100 – x)/2 = 50 – 0.5x. Thus, A(x) = x(50 – 0.5x) = $50x - 0.5x^2$.

Piecewise functions.

Some situations can only be described by more than one formula, depending on the value of the independent variable.

Example 7. A user is charged P300 monthly for a particular mobile plan, which includes 100 free text messages. Messages in excess of 100 are charged P1 each. Represent the monthly cost for text messaging using the function t(*m*), where *m* is the number of messages sent in a month.

Solution. The cost of text messaging can be expressed by the piecewise function:

$$t(m) = \begin{cases} 300 & , if \;\; 0 < m \leq 100 \\ 300 + m & , if \;\; m > 100 \end{cases}$$

Example 8. A jeepney ride costs P8.00 for the first 4 kilometers, and each additional integer kilometer adds P1.50 to the fare. Use a piecewise function to represent the jeepney fare in terms of the distance *(d)* in kilometers.

Solution. The input value is distance and the output is the cost of the jeepney fare. If *F(d)* represents the fare as a function of distance, the function can be represented as follows:

$$F(d) = \begin{cases} 8.00 & if \;\; 0 < d \leq 4 \\ (8 + 1.5\lfloor d \rfloor) & if \;\; d > 4 \end{cases}$$

Note that $\lfloor d \rfloor$is the floor function applied to d. The floor function gives the largest integer less than or equal to *d*, e.g. $\lfloor 4.1 \rfloor = \lfloor 4.9 \rfloor = 4$.

Example 9. Water can exist in three states: solid ice, liquid water, and gaseous water vapor. As ice is heated, its temperature rises until it hits the melting point of 0°C and stays constant until the ice melts. The temperature then rises until it hits the boiling point of 100°C and stays constant until the water evaporates. When the water is in a gaseous state, its temperature can rise above 100°C (This is why steam can cause third degree burns!).

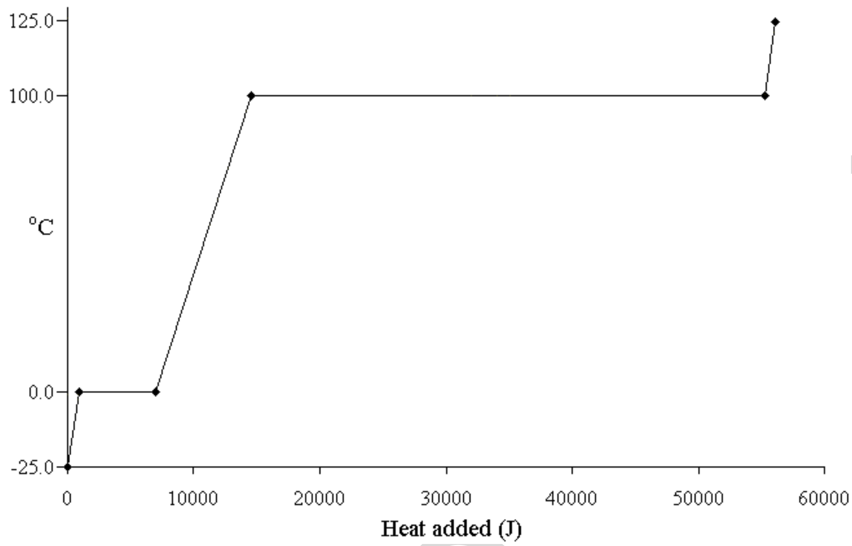
A solid block of ice is at -25°C and heat is added until it completely turns into water vapor. Sketch the graph of the function representing the temperature of water as a function of the amount of heat added in Joules given the following information:

- The ice reaches 0°C after applying 940 J.
- The ice completely melts into liquid water after applying a total of 6,950 J.

- The water starts to boil (100°C) after a total of 14,470 J.
- The water completely evaporates into steam after a total of 55,260 J.

Assume that rising temperature is linear. Explain why this is a piecewise function.

Solution. Let T(x) represent the temperature of the water in degrees Celsius as a function of cumulative heat added in Joules. The function T(x) can be graphed as follows:



This is a piecewise function because the temperature rise can be expressed as a linear function with positive slope until the temperature hits 0°C, then it becomes a constant function until the total heat reaches 6,950K J. It then becomes linear again until the temperature reaches 100°C, and becomes a constant function again until the total heat reaches 55,260 J.

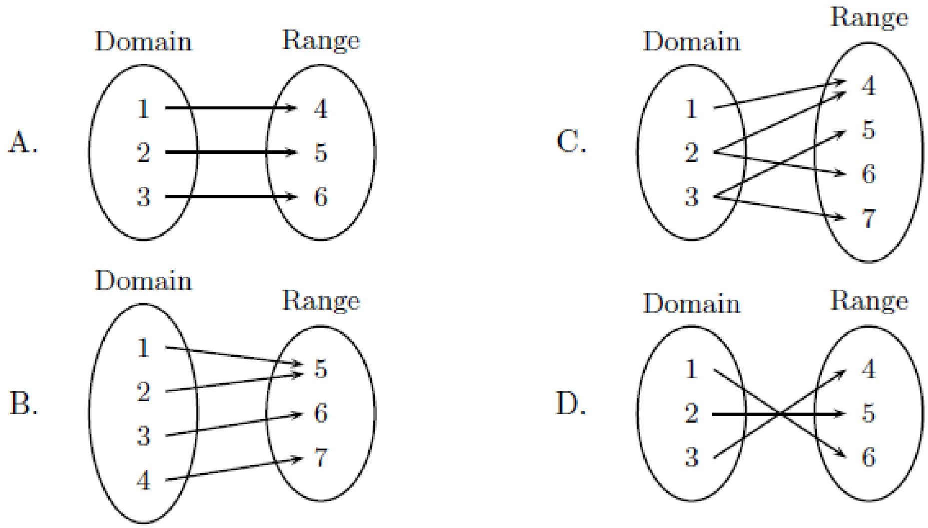
Solved Examples

1. Is the relation {(0,0), (1,1), (2,4), (3,9), … (n, n^2), …} a function?

Solution.

Yes, it is a function.

2. Which of the following diagram represents a relation that is NOT a function?



***Solution*.**

C. All diagrams, except for C, represent a function.

3. Can the graph of a circle be considered a function?

***Solution*.**

No, it cannot. A circle will fail the vertical line test.

4. Give the domain of $y = \sqrt{2 - x}$ using set builder notation.

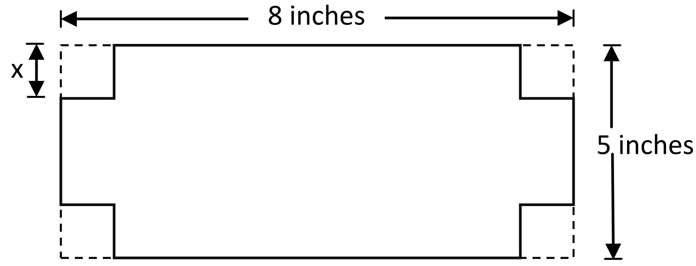
Solution.

$\{x : x \in \mathbb{R}, x \leq 2\}$

5. Contaminated water is subjected to a cleaning process. The concentration of pollutants is initially 10 mg per liter of water. If the cleaning process can reduce the pollutant by 5% each hour, define a function that can represent the concentration of pollutants in the water in terms of the number of hours that the cleaning process has taken place.

Solution. After 1 hour, the concentration of pollutants is (10)*(0.95). After 2 hours, it is this value, times 0.95, or [(10)*(0.95)](0.95) = $10(0.95)^2$. In general, after t hours, the concentration is $C(t) = (10)(0.95)^t$ mg per liter of water.

6. Squares of side *x* are cut from each corner of an 8 in x 5 in rectangle (see figure), so that its sides can be folded to make a box with no top. Define a function in terms of x that can represent the volume of this box.



Solution. The length and width of the box are 8 – 2x and 5 – 2x, respectively. Its height is x. Thus, the volume of the box can be represented by the function

$$V(x) = (8 - 2x)(5 - 2x)x = 40x - 26x^2 + 4x^3.$$

7. A certain chocolate bar costs P35.00 per piece. However, if you buy more than 10 pieces, they will be marked down to a price of P32.00 per piece. Use a piecewise function to represent the cost in terms of the number of chocolate bars bought.

Solution.

$$f(n) = \begin{cases} 35n & , if\ \ 0 < n \leq 10 \\ 32n & , if\ n > 10 \end{cases}$$

8. A school's fair committee wants to sell t-shirts for their school fair. They found a supplier that sells t-shirts at a price of P175.00 a piece but can charge P15,000 for a bulk order of 100 shirts and P125.00 for each excess t-shirt after that. Use a piecewise function to represent the cost in terms of the number of t-shirts purchased.

Solution.

$$F(n) = \begin{cases} 175n & , if\ \ 0 < n \leq 99 \\ 15\,000 & , if\ n = 100 \\ 15\,000 + 125(n - 100), & if\ n > 100 \end{cases}$$

9. The fee to park in the parking lot of a shopping mall costs P40.00 for the first two hours and an extra P10.00 for each hour (or a fraction of it) after that. If you park for more than twelve hours, you instead pay a flat rate of P200.00. Represent your parking fee using the function p(*t*) where *t* is the number of hours you parked in the mall.

Solution.

$$p(t) = \begin{cases} 40 & ,if\ \ 0 < t \leq 2 \\ 40 + 10\lceil t-2 \rceil & ,if\ 2 < t \leq 12 \\ 200 & ,if\ t > 12 \end{cases}$$

Here $\lceil t-2 \rceil$ is the ceiling function applied to t – 2. The ceiling function of a number x gives the smallest integer greater than or equal to x,

e.g. $\lceil 5.1 \rceil = \lceil 5.8 \rceil = \lceil 6 \rceil = 6$.

Lesson 1 Supplementary Exercises

1. For which values of k is the set of order pairs $\{(2,4),(k,6),(4,k)\}$ a function?

2. Which of the following statements represents a function?
   (a) Students to their current age.
   (b) Countries to its capital.
   (c) A store to its merchandise.

3. Which of the following letters will pass the vertical line test? V W X Y Z

4. Give the domain of $y = \frac{1}{\sqrt{x^2-4}}$ in set builder notation.

5. A person is earning P600 per day to do a certain job. Express the total salary *S* as a function of the number *n* of days that the person works.

6. A canned drink will be made using 40 in^2 of aluminum. Let r be the radius of the can and let h be the height. Define a function in terms of *r* that can represent the volume of the can.

7. A computer shop charges 20 pesos per hour (or a fraction of an hour) for the first two hours and an additional 10 pesos per hour for each succeeding hour. Represent your computer rental fee using the function R(*t*) where *t* is the number of hours you spent on the computer.

8. A taxi ride costs P40.00 for the first 500 meters, and each additional 300 meters (or a fraction thereof) adds P3.50 to the fare. Use a piecewise function to represent the taxi fare in terms of the distance d in meters.

9. Temperature readings T (in ^{0} C) were recorded every three hours from midnight until 6 PM. The time *t* was measured in hours from midnight.

T	0	3	6	9	12	15	18
T	24	26	28	30	32	30	28

   (a) Use the data to sketch a rough graph of T as a function of *t*.
   (b) Assuming that the peak temperature was recorded during 12 noon, what do you think is the temperature by 9 PM?

Lesson 2: Evaluating Functions

Learning Outcome(s): At the end of the lesson, the learner is able to evaluate functions and solve problems involving functions.
.

Lesson Outline:

1. Evaluating functions

Evaluating a function means replacing the variable in the function, in this case *x*, with a value from the function's domain and computing for the result. To denote that we are evaluating *f* at *a* for some *a* in the domain of *f*, we write *f(a)*.

Example 1. Evaluate the following functions at *x* = *1.5*:

(g) $f(x) = 2x + 1$
(h) $q(x) = x^2 - 2x + 2$
(i) $g(x) = \sqrt{x + 1}$
(j) $r(x) = \frac{2x+1}{x-1}$
(k) $F(x) = \lfloor x \rfloor + 1$, where $\lfloor x \rfloor$ is the greatest integer function.

Solution. Substituting 1.5 for x in the functions above, we have

(a) $f(1.5) = 2(1.5) + 1 = 4$
(b) $q(1.5) = (1.5)^2 - 2(1.5) + 2 = 2.25 - 3 + 2 = 1.25$
(c) $g(1.5) = \sqrt{1.5 + 1} = \sqrt{2.5}$
(d) $r(1.5) = \frac{2(1.5)+1}{(1.5)-1} = \frac{3+1}{0.5} = 8$
(e) $F(1.5) = \lfloor 1.5 \rfloor + 1 = 1 + 1 = 2$

Example 2. Find g(−4) and r(1) where g and r are as defined in the previous example.

Solution. This is not possible because −4 is not in the domain of g(x) and 1 is not in the domain of r(x).

Example 3. Evaluate the following functions, where f and q are as defined in Example 1.

$(a) f(3x - 1)$ $(b)\ q(2x + 3)$

Solution.

$$(a)\ f(3x - 1) = 2(3x - 1) + 1 = 6x - 2 + 1 = 6x - 1$$

$$(b)\ q(2x + 3) = (2x + 3)^2 - 2(2x + 3) + 2 = (4x^2 + 12x + 9) - 4x - 6 + 2$$
$$= 4x^2 + 8x + 5$$

Solved Examples

1. Evaluate the following functions at x=3.

(a) $f(x) = x - 3$
(b) $g(x) = x^2 - 3x + 5$
(c) $h(x) = \sqrt[3]{x^3 + x + 3}$
(d) $p(x) = \frac{x^2+1}{x-4}$
(e) $f(x) = |x - 5|$ where |x – 5| means the absolute value of x – 5.

Solution.

(a) $f(3) = 3 - 3 = 0$
(b) $g(3) = (3)^2 - 3(3) + 5 = 9 - 9 + 5 = 5$
(c) $h(3) = \sqrt[3]{(3)^3 + 3 + 3} = \sqrt[3]{27 + 6} = \sqrt[3]{33}$
(d) $p(3) = \frac{(3)^2+1}{3-4} = \frac{10}{-1} = -10$
(e) $f(3) = |3 - 5| = |-2| = 2$

2. For what values of x can we not evaluate the function $f(x) = \frac{x+3}{x^2-4}$?

Solution.

The domain of the function is given by $\{x : x \in \mathbb{R}, x \neq \pm 2\}$. Since 2 and -2 are not in the domain, we cannot evaluate the function at x = -2,2.

3. Evaluate $f(a + b)$ where $f(x) = 4x^2 - 3x$.

Solution.

$$f(a + b) = 4(a + b)^2 - 3(a + b) = 4(a^2 + 2ab + b^2) - 3a - 3b$$
$$= 4a^2 - 3a + 8ab - 3b + 4b^2$$

4. Suppose that s(T) is the top speed (in km per hour) of a runner when the temperature is T degrees Celsius. Explain what the statements s(15) = 12 and s(30) = 10 mean.

Solution. The first equation means that when the temperature is 15°C, then the top speed of a runner is 12 km per hour. However, when temperature rises to 30°C, the top speed is reduced to 10 km per hour.

5. The velocity V (in m/s) of a ball thrown upward t seconds after the ball was thrown is given by V(t) = 20 – 9.8t. Calculate V(0) and V(1), and explain what these results mean.

Solution. V(0) = 20 – 9.8(0) = 20 and V(1) = 20 – 9.8(1) = 10.2. These results indicate that the initial velocity of the ball is 20 m/s. After 1 second, the ball is traveling more slowly, at 10.2 m/s.

Lesson 2 Supplementary Exercises

1. Evaluate the following functions at $x = -4$.
   (a) $f(x) = x^3 - 64$
   (b) $g(x) = |x^3 - 3x^2 + 3x - 1|$
   (c) $r(x) = \sqrt{5 - x}$
   (d) $q(x) = \frac{x+3}{x^2+7x+12}$

2. Given $f(x) = \begin{cases} 9 - x^2\ , & x < 2 \\ \sqrt{x+7}, & 2 \leq x < 10 \\ \lceil x - 4 \rceil & , x \geq 10 \end{cases}$, give the values of the following:
   (a) $f(2)$
   (b) $f(12.5)$
   (c) $f(-3)$
   (d) $f(5)$
   (e) $f(1.5)$

3. Given $f(x) = x^2 - 4x + 4$. Solve for
   (a) $f(3)$
   (b) $f(x + 3)$
   Is $f(x + 3)$ the same as $f(x) + f(3)$?

4. A computer shop charges P20.00 per hour (or a fraction of an hour) for the first two hours and an additional P10.00 per hour for each succeeding hour. Find how much you would pay if you used one of their computers for:
   (a) 40 minutes
   (b) 3 hours
   (c) 150 minutes

5. Under certain circumstances, a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + 15(2.1)^{-0.3t}}\ ,$$

where p(t) is the proportion of the population that knows the rumor (t) days after the rumor started. Find p(4) and p(10), and interpret the results.

Lesson 3: Operations on Functions

Learning Outcome(s): At the end of the lesson, the learner is able to perform addition, subtraction, multiplication, division, composition of functions, and solve problems involving functions.

Lesson Outline:

1. Review: Operations on algebraic expressions
2. Addition, subtraction, multiplication, and division of functions
3. Function composition

Addition and Subtraction:
(a) Find the least common denominator (LCD) of both fractions.
(b) Rewrite the fractions as equivalent fractions with the same LCD.
(c) The LCD is the denominator of the resulting fraction.
(d) The sum or difference of the numerators is the numerator of the resulting fraction.

Example 1. Find the sum of $\frac{1}{3}$ and $\frac{2}{5}$.

Solution. The LCD of the two fractions is 15.

$$\frac{1}{3}+\frac{2}{5}=\frac{5}{15}+\frac{6}{15}=\frac{5+6}{15}=\frac{11}{15}$$

Example 2. Find the sum of $\frac{1}{x-3}$ and $\frac{2}{x-5}$.

Solution. The LCD of the two fractions is $(x-3)(x-5)$ or $x^2-8x+15$.

$$\frac{1}{x-3}+\frac{2}{x-5}=\frac{x-5}{x^2-8x+15}+\frac{2(x-3)}{x^2-8x+15}=\frac{x-5+2x-6}{x^2-8x+15}=\frac{3x-11}{x^2-8x+15}$$

Multiplication:
(a) Rewrite the numerator and denominator in terms of its prime factors.
(b) Common factors in the numerator and denominator can be simplified as "1" (this is often called "cancelling").
(c) Multiply the numerators together to get the new numerator.
(d) Multiply the denominators together to get the new denominator.

Example 3. Find the product of $\frac{10}{21}$ and $\frac{15}{8}$. Use cancellation of factors when convenient.

Solution. Express the numerators and denominators of the two fractions into their prime factors. Multiply and cancel out common factors in the numerator and the denominator to reduce the final answer to lowest terms.

$$\frac{10}{21}\cdot\frac{15}{8}=\frac{2\cdot 5}{3\cdot 7}\cdot\frac{3\cdot 5}{2\cdot 2\cdot 2}=\frac{2\cdot 5\cdot 3\cdot 5}{3\cdot 7\cdot 2\cdot 2\cdot 2}=\frac{25}{28}$$

Example 4. Find the product of $\frac{x^2-4x-5}{x^2-3x+2}$ and $\frac{x^2-5x+6}{x^2-3x-10}$.

Solution. Express the numerators and denominators of the two rational expressions into their prime factors. Multiply and cancel out common factors in the numerator and the denominator to reduce the final answer to lowest terms. Note the similarity in the process between this example and the previous one on fractions.

$$\frac{x^2-4x-5}{x^2-3x+2}\cdot\frac{x^2-5x+6}{x^2-3x-10}=\frac{(x+1)(x-5)}{(x-2)(x-1)}\cdot\frac{(x-2)(x-3)}{(x-5)(x+2)}$$
$$=\frac{(x+1)\cancel{(x-5)}\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x-1)\cancel{(x-5)}(x+2)}$$
$$=\frac{(x+1)(x-3)}{(x-1)(x+2)}$$
$$=\frac{x^2-2x-3}{x^2+x-2}$$

.

Division:

To divide two fractions or rational expressions, multiply the dividend with the reciprocal of the divisor.

Example 5. Divide $\frac{2x^2+x-6}{2x^2+7x+5}$ by $\frac{x^2-2x-8}{2x^2-3x-20}$**.**

$$\frac{2x^2+x-6}{2x^2+7x+5}\div\frac{x^2-2x-8}{2x^2-3x-20}=\frac{2x^2+x-6}{2x^2+7x+5}\cdot\frac{2x^2-3x-20}{x^2-2x-8}$$
$$=\frac{(2x-3)(x+2)}{(2x+5)(x+1)}\cdot\frac{(x-4)(2x+5)}{(x+2)(x-4)}$$
$$=\frac{(2x-3)\cancel{(x+2)}\cancel{(x-4)}\cancel{(2x+5)}}{\cancel{(2x+5)}(x+1)\cancel{(x+2)}\cancel{(x-4)}}$$
$$=\frac{2x-3}{x+1}$$

Definition. Let *f* and *g* be functions.

1. Their sum, denoted by $f+g$, is the function denoted by $(f+g)(x)=f(x)+g(x)$.
2. Their difference, denoted by $f-g$, is the function denoted by $(f-g)(x)=f(x)-g(x)$.
3. Their product, denoted by $f\cdot g$, is the function denoted by $(f\cdot g)(x)=f(x)\cdot g(x)$.
4. Their quotient, denoted by f/g , is the function denoted by $(f/g)(x)=f(x)/g(x)$, excluding the values of x where $g(x)=0$.

Use the following functions below for Example 5.

- $f(x) = x + 3$
- $p(x) = 2x - 7$
- $v(x) = x^2 + 5x + 4$
- $g(x) = x^2 + 2x - 8$
- $h(x) = \frac{x+7}{2-x}$
- $t(x) = \frac{x-2}{x+3}$

Example 5. Determine the following functions.

(a) $(v + g)(x)$
(b) $(f \cdot p)(x)$
(c) $(f + h)(x)$
(d) $(p - f)(x)$
(e) $(v/g)(x)$

Solution.

(a) $(v + g)(x) = (x^2 + 5x + 4) + (x^2 + 2x - 8)$
$= x^2 + 5x + 4 + x^2 + 2x - 8 = 2x^2 + 7x - 4$

(b) $(f \cdot p)(x) = (x + 3)(2x - 7) = 2x^2 - x - 21$

(c) $(f + h)(x) = (x + 3) + \frac{x+7}{2-x} = (x + 3) \cdot \frac{2-x}{2-x} + \frac{x+7}{2-x} = \frac{(x+3)(2-x)+(x+7)}{2-x}$
$= \frac{6 - x - x^2 + x + 7}{2 - x} = \frac{13 - x^2}{2 - x} = \frac{x^2 - 13}{x - 2}$

(d) $(p - f)(x) = (2x - 7) - (x + 3) = 2x - 7 - x - 3 = x - 10$

(e) $\left(\frac{v}{g}\right)(x) = (x^2 + 5x + 4) \div (x^2 + 2x - 8) = \frac{x^2+5x+4}{x^2+2x-8}$

Use the following functions for Examples 6-12:

- $f(x) = 2x + 1$
- $q(x) = x^2 - 2x + 2$
- $r(x) = \frac{2x+1}{x-1}$

Example 6. Express the function $f_1(x) = x^2 + 3$ as a sum or difference of the functions above.

Solution.

The solution can involve some trial and error. Add $q(x)$ and $f(x)$ and check if the sum is $x^2 + 3$.

$$q(x) + f(x) = (x^2 - 2x + 2) + (2x + 1) = x^2 + 3 = f_1(x)$$

Thus $f_1(x) = q(x) + f(x)$.

Example 7. Express the function $f_2(x) = x^2 - 4x + 1$ as the sum or difference of the given functions.

Solution.

Again, the solution can involve trial and error. Check if $q(x) - f(x)$ gives $x^2 - 4x + 1$.

$$q(x) - f(x) = (x^2 - 2x + 2) - (2x + 1) = x^2 - 4x + 1 = f_2(x)$$

Thus $f_2(x) = q(x) - f(x)$.

Example 8. Express the function $f_3(x) = \frac{2x^2+x}{x-1}$ as a sum or difference of the given functions.

Solution.

Because $\frac{2x^2+x}{x-1}$ has $(x-1)$ as a denominator, then $r(x) = \frac{2x+1}{x-1}$ must be one of the functions in our solution. Let us try to add $f(x)$ and $r(x)$:

$$f(x) + r(x) = 2x + 1 + \frac{2x+1}{x-1} = \frac{(2x+1)(x-1)}{x-1} + \frac{2x+1}{x-1} = \frac{(2x+1)(x-1)+(2x+1)}{x-1}$$

$$= \frac{(2x^2 - x - 1) + (2x + 1)}{x - 1} = \frac{2x^2 + x}{x - 1} = f_3(x)$$

Example 9. Write down the answers from the previous items in the notation denoted in the definitions.

Solution.

(a) $f_1(x) = q(x) + f(x) = (q + f)(x)$
(b) $f_2(x) = q(x) - f(x) = (q - f)(x)$
(c) $f_3(x) = f(x) + r(x) = (f + r)(x)$

Example 10. Express the function $g_1(x) = 2x^3 - 3x^2 + 2x + 2$ as a product or quotient of the given functions.

Solution.

Since $2x^3 - 3x^2 + 2x + 2$ is cubic, then it is possibly the product of $f(x)$ and $q(x)$.

$$\begin{aligned} f(x) \cdot q(x) &= (2x + 1)(x^2 - 2x + 2) = (2x)(x^2 - 2x + 2) + (x^2 - 2x + 2) \\ &= (2x^3 - 4x^2 + 4x) + (x^2 - 2x + 2) = 2x^3 - 3x^2 + 2x + 2 = g_3(x) \end{aligned}$$

Thus, $g_3(x) = f(x) \cdot q(x)$.

Example 11. Express the function $g_2(x) = x - 1$ as a product or quotient of the given functions.

Solution.

The function $r(x) = \frac{2x+1}{x-1}$ involves $x - 1$. The goal is to "get rid" of $2x + 1$. This can be done by dividing $f(x)$ by $r(x)$:

$$\frac{f(x)}{r(x)} = (2x+1) \div \frac{2x+1}{x-1} = (2x+1) \cdot \frac{x-1}{2x+1} = \frac{2x+1}{2x+1} \cdot (x-1) = x - 1 = g_2(x)$$

Thus, $g_2(x) = f(x)/r(x)$.

Example 12. Express the function $g_3(x) = \frac{1}{x-1}$ as a product or quotient of the given functions.

Solution.

The function $g_3(x) = \frac{1}{x-1}$ is very similar to $r(x) = \frac{2x+1}{x-1}$. The goal is to "get rid" of $2x + 1$. This can be done by dividing $r(x)$ by $f(x) = 2x + 1$.

$$\frac{r(x)}{f(x)} = \frac{2x+1}{x-1} \div (2x+1) = \frac{2x+1}{x-1} \cdot \frac{1}{2x+1} = \frac{1}{x-1} = g_3(x)$$

Thus, $g_3(x) = r(x)/f(x)$.

Definition. Let *f* and *g* be functions. The **composite function** denoted by $(f \circ g)$ is defined by $(f \circ g)(x) = f(g(x))$. The process of obtaining a composite function is called **function composition.**

For examples 13 to 16, use the following functions:

$f(x) = 2x + 1$ $\qquad$ $g(x) = \sqrt{x+1}$ $\qquad$ $p(x) = \frac{2x+1}{x-1}$

$q(x) = x^2 - 2x + 2$ $\qquad$ $F(x) = \lfloor x \rfloor + 1.$

Example 13. Find and simplify $(g \circ f)(x)$.

Solution.

$$(g \circ f)(x) = g(f(x)) = \sqrt{f(x)+1} = \sqrt{(2x+1)+1} = \sqrt{2x+2}$$

Example 14. Find and simplify $(q \circ f)(x)$.

Solution.

$$(q \circ f)(x) = q(f(x)) = [f(x)]^2 - 2[f(x)] + 2 = (2x+1)^2 - 2(2x+1) + 2$$
$$= (4x^2 + 4x + 1) - (4x + 2) + 2 = 4x^2 + 1$$

Example 15. Find and simplify $(f \circ p)(x)$.

Solution.

$$(f \circ p)(x) = f(p(x)) = 2p(x) + 1 = 2\left[\frac{2x+1}{x-1}\right] + 1 = \frac{4x+2}{x-1} + 1$$
$$= \frac{(4x+2)+(x-1)}{x-1} = \frac{5x+1}{x-1}$$

Example 16. Find and simplify $(F \circ p)(5)$.

Solution.

$$(F \circ p)(5) = F(p(5)) = \lfloor p(5) \rfloor + 1 = \left\lfloor \frac{2(5)+1}{5-1} \right\rfloor + 1 = \left\lfloor \frac{11}{4} \right\rfloor + 1 = 2 + 1 = 3$$

Solved Examples

1. Let $f(x) = 3x^2 - 2x - 1$, $g(x) = x^2 - 1$, and $h(x) = f(x) + g(x)$. Find:
   (a) $(f - g)(x)$
   (b) $f(-1) \cdot g(2) \cdot h(0)$
   (c) $\frac{g(x)}{f(x)}$
   (d) $h(x - 1)$
   (e) $f(3) + g(2)$

***Solution*.**

(a) $(f - g)(x) = (3x^2 - 2x - 1) - (x^2 - 1) = 2x^2 - 2x$
(b) $f(-1) \cdot g(2) \cdot h(0)$
$= [3(-1)^2 - 2(-1) - 1] \cdot [(2)^2 - 1] \cdot [3(0)^2 - 2(0) - 1 + (0)^2 - 1]$
$= 4 \cdot 3 \cdot (-2) = -24$
(c) $\frac{g(x)}{f(x)} = \frac{x^2-1}{3x^2-2x-1} = \frac{(x+1)\cancel{(x-1)}}{(3x+1)\cancel{(x-1)}} = \frac{x+1}{3x+1}$
(d) $h(x-1) = 3(x-1)^2 - 2(x-1) - 1 + (x-1)^2 - 1$
$= 3(x^2 - 2x + 1) - 2x + 2 - 1 + (x^2 - 2x + 1) - 1 = 4x^2 - 10x + 4$
(e) $f(3) + g(2) = [3(3)^2 - 2(3) - 1] + [(2)^2 - 1]$
$= (27 - 6 - 1) = 20 + 3 = 23$

2. Let $f(x) = 2x - 15, g(x) = x^2 + 19x + 90, h(x) = \sqrt{x+6}$. Find:

   (a) $(g \circ f)(x)$
   (b) $(h \circ g)(-4)$
   (c) $f(f(f(f(15))))$

Solution.

(a) $(g \circ f)(x) = (2x - 15)^2 + 19(2x - 15) + 90$
$= 4x^2 - 60x + 225 + 38x - 285 + 90 = 4x^2 - 22x + 30$

(b) $(h \circ g)(-4) = h((-4)^2 + 19(-4) + 90) = h(16 - 76 + 90) = h(30) = \sqrt{30 + 6} = \sqrt{36} = 6$

(c) $f(f(f(f(15)))) = f\left(f(f(15))\right) = f(f(15)) = f(15) = 15$

3. Express the following functions as the sum, difference, quotient, or product of $f(x) = 3x^2 - 2x - 1$ and $g(x) = x^2 - 1$.

(a) $t(x) = -2x^2 + 2x$
(b) $u(x) = 8x^2 - 4x - 4$
(c) $v(x) = 2x - 2$
(d) $w(x) = \frac{-3x-1}{x+1}$

Solution.

(a) $t(x) = -2x^2 + 2x = (x^2 - 1) - (3x^2 - 2x - 1) = g(x) - f(x)$

(b) $u(x) = 8x^2 - 4x - 4 = (6x^2 - 4x - 2) + (2x^2 - 2)$
$= 2[(3x^2 - 2x - 1) + (x^2 - 1)] = 2(f + g)(x) = 2f(x) + 2g(x)$

(c $v(x) = 2x - 2 = (3x^2 - 3) - (3x^2 - 2x - 1) = 3(x^2 - 1) - (3x^2 - 2x - 1)$
$= 3g(x) - f(x)$

(d) $w(x) = \frac{-3x-1}{x+1} = \frac{-(3x+1)}{x+1} = \frac{-(3x+1)(x-1)}{(x+1)(x-1)} = \frac{-(3x^2-2x-1)}{x^2-1} = \frac{-f(x)}{g(x)}$

4. Suppose that N(x) = x denotes the number of shirts sold by a shop, and the selling price per shirt is given by p(x) = 250 – 5x, for $0 \leq x \leq 20$. Find (N · p)(x) and describe what it represents.

Solution. (N · p)(x) = N(x) · p(x) = x(250 – 5x) = 250x – $5x^2$ ($0 \leq x \leq 20$). Since this function is the product of the quantity sold and the selling price, then (N · p)(x) represents the revenue earned by the company.

5. A spherical balloon is being inflated. Let r(t) = 3t cm represent its radius at time *t* seconds, and let *g(r)* = $\frac{4}{3}\pi r^3$ be the volume of the same balloon if its radius is r. Write (g ∘ r) in terms of t, and describe what it represents.

Solution. (g ∘ r) = g(r(t)) = $\frac{4}{3}\pi(3t)^3 = 36\pi t^3$. This function represents the volume of the balloon at time *t* seconds.

Lesson 3 Supplementary Exercises

1. Let *f* and *g* be defined as $f(x) = x - 5$ and $g(x) = x^2 - 1$. Find
(a) $f + g$
(b) $f - g$
(c) $f \cdot g$
(d) f/g
(e) g/f

2. Let *f* and *g* be defined as in the previous example. Express the following functions as the sum, difference, quotient, or product of f and g.

(a) $P(x) = x^2 - x - 4$
(b) $Q(x) = -x^2 + 2x - 9$
(c) $R(x) = \frac{x-5}{2x^2-2}$
(d) $S(x) = \frac{x^2-x+4}{x-5}$
(e) $T(x) = \frac{x^3-4x^2-11x+30}{x^2-1}$

3. Let $f(x) = x^2 - 1$ and $g(x) = \frac{1}{x}$. Find

(a) $(f\text{o}g)(x)$
(b) $(g\text{o}f)(-1)$
(c) $(f\text{o}f)(x)$
(d) $(g\text{o}g)(5)$

4. Given the following, find $f(x)$.

(a) $(f + g)(x) = 2x^2 - 5x + 6, g(x) = x^2 + 8x - 7$
(b) $(f \cdot \text{g})(\text{x}) = x^4 + 16x^2 + 256, g(x) = x^2 + 4x + 16$
(c) $(f\text{o}g)(x) = (3x - 2)^4, g(x) = 3x - 2$
(d) $(g\text{o}f)(x) = 18x - 25. g(x) = 6x - 7$

5. Suppose that N(x) = x denotes the number of bags sold by a shop, and the selling price per bag is given by p(x) = 320 – 8x, for 0 ≤ x ≤ 10. Suppose further that the cost of producing x bags is given by C(x) = 200x. Find (a) (N · p)(x) and (b) (N · p – C)(x). What do these functions represent?

6. Let x represent the regular price of a book.

a) Give a function f that represents the price of the book if a P100 price reduction applies.

b) Give a function g that represents the price of the book if a 10% discount applies.

c) Compute (f ∘ g)(x) and (g ∘ f)(x). Describe what these mean. Which of these give a better deal for the customer?

Lesson 4: Representing Real-Life Situations Using Rational Functions

Learning Outcome(s): At the end of the lesson, the learner is able to represent real-life situations rational functions.

Lesson Outline:

1. Review: Polynomial functions
2. Rational functions and real-life situations

Recall the definition of a polynomial function.

Definition: A **polynomial function p of degree n** is a function that can be written in the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$, $a_n \neq 0$, and n is a positive integer. Each addend of the sum is a **term** of the polynomial function. The constants $a_0, a_1, a_2, \ldots, a_n$ are the **coefficients**. The **leading coefficient** is a_n. The **leading term** is $a_n x^n$, and the **constant term** is a_0.

Definition: A **rational function** is a function of the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not the zero function (i.e., $q(x) \not\equiv 0$). The domain of $f(x)$ is the set of all values of x where $q(x) \neq 0$.

Example 1. An object is to travel a distance of 10 meters. Express velocity v as a function of travel time t, in seconds.

Solution. The following table of values show v for various values of t.

t (seconds)	1	2	4	5	10
v (meters per second)	10	5	2.5	2	1

The function $v(t) = \frac{10}{t}$ can represent v as a function of t.

Example 2. Suppose that $c(t) = \frac{5t}{t^2+1}$ (in mg/mL) represents the concentration of a drug in a patient's bloodstream t hours after the drug was administered. Construct a table of values for c(t) for t = 1, 2, 5, 10. Round off answers to three decimal places. Use the table to sketch a graph and interpret the results.

Solution.

t	0	1	2	5	10
c(t)	0	2.5	3	0.962	0.495

The graph indicates that the maximum drug concentration occurs around 1 hour after the drug was administered (calculus can be used to determine the exact value at which the maximum occurs). After 1 hour, the graph suggests that drug concentration decreases until it is almost zero.

Solved Examples

1. In an organ pipe, the frequency f of vibration of air is inversely proportional to the length L of the pipe.[1] Suppose that the frequency of vibration in a 10-foot pipe is 54 vibrations per second. Express f as a function of L.

   ***Solution*.**

Since f is inversely proportional to L, then $f = \frac{k}{L}$, where k is the constant of proportionality.

If L = 10 then f = 54. Thus, $54 = \frac{k}{10} \Longrightarrow k = 540$. Thus, the function $f(L) = \frac{540}{L}$ represents f as a function of L.

2. The distance from Manila to Baguio is around 250 kilometers.

   (a) How long will it take you to get to Baguio if your average speed is 25 kilometers per hour? 40 kilometers per hour? 50 kilometers per hour?
   (b) Construct a function (s) , where s is the speed of travel, that describes the time it takes to drive from Manila to Baguio.

[1] Barnett, R.A., Ziegler, M.R., Byleen, K.E., &Sobecki, D. (2008). *Precalculus*(7th ed). New York: McGraw Hill.

***Solution*.**

(a) Distance is calculated as the product of speed and time. So we can get the time by dividing distance by the speed.

250 kilometers/ 25 kilometers per hour = 10 hours
250 kilometers/ 40 kilometers per hour = 6.25 hours
250 kilometers/ 50 kilometers per hour = 5 hours

(b) Since time is the quotient of distance and speed, we can write out the function as

$$t(s) = \frac{d}{s}$$

The distance is fixed at 250 kilometers so the final function we have is

$$t(s) = \frac{250}{s}$$

Lesson 4 Supplementary Exercises

1. Given the polynomial function $p(x) = 12 + 4x - 3x^2 - x^3$, find
   (a) The degree of the polynomial
   (b) The leading coefficient
   (c) The constant term
   (d) The number of zeroes

2. The budget of a university organization is split evenly among its various committees. If they have a budget of P60,000:
   (a) Construct a function $M(n)$ which would give the amount of money each of the n number of committees would receive.
   (b) If the organization has eight committees, how much would each committee have?

3. A company has a budget of P90,000 to be split evenly among its various offices. The marketing office of the company receives twice the amount of money than the other offices.
   (a) Given x as the number of offices in the company, construct a function $f(x)$ which would give the amount of money each of the non-marketing offices would receive.
   (b) If the company had five offices, how much would the marketing office receive? How much would each of the non-marketing offices receive?

4. Let $C(t) = \frac{4t}{t^2+4}$ be the function that describes the concentration of a certain medication in the bloodstream over time t.
   (a) What is $C(0)$? Why is that so?
   (b) Construct a table of values for when t is equal to 0,1,2,3,4, and 5.
   (c) Interpret your answers in relation to drug concentration.

Lesson 5: Rational Functions, Equations, and Inequalities

Learning Outcome(s): At the end of the lesson, the learner is able to distinguish among rational functions, rational equations, and rational inequalities

Lesson Outline:

1. Rational functions, rational equations, and rational inequalities

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Definition: A **rational expression** is an expression that can be written as a ratio of two polynomials.

Some examples of rational expressions are $\frac{2}{x}, \frac{x^2+2x+3}{x+1}, and\ \frac{5}{x-3}$.

The definitions of rational equations, inequalities, and functions are shown below.

		Rational Inequality	Rational Function
Definition	An equation involving rational expressions.	An inequality involving rational expressions.	A function of the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not the zero function (i.e., $q(x) \not\equiv 0$).
Example	$\frac{2}{x} - \frac{3}{2x} = \frac{1}{5}$	$\frac{5}{x-3} \leq \frac{2}{x}$	$f(x) = \frac{x^2+2x+3}{x+1}$

A rational equation or inequality can be solved for all x values that satisfy the equation or inequality. A rational function expresses a relationship between two variables (such as x and y), and can be represented by a table of values or a graph (Lessons 6-7).

Solved Examples

Determine whether the given is a rational function, a rational equation, a rational inequality or none of these.

1. $\frac{2+x}{x-1} = 8$ (Answer: Rational equation)
2. $x > \sqrt{x+2}$ (Answer: None of these)
3. f(x) = $6 - \frac{x+3}{x^2-5}$ (Answer: Rational Function)
4. $2x \geq \frac{7}{x+4}$ (Answer: Rational Inequality)
5. $\frac{x}{2} = \frac{4}{x+9x^3}$ (Answer: Rational equation)

Lesson 5 Supplementary Exercises

Determine whether the given is a rational function, a rational equation, a rational inequality or none of these.

1. $y = 5x^3 - 2x + 1$
2. $\frac{8}{x} - 8 = \frac{x}{2x-1}$
3. $\sqrt{x-2} = 4$
4. $\frac{x-1}{x+1} = x^2$
5. $g(x) = \frac{7x^3 - 4\sqrt{x} + 1}{x^2+3}$
6. $6x - \frac{5}{x+3} \geq 0$

Lesson 6: Solving Rational Equations and Inequalities

Learning Outcome(s): At the end of the lesson, the learner is able to solve rational equations and inequalities, and solve problems involving rational equations and inequalities.

Lesson Outline:

1. Solving rational equations.
2. Solving rational inequalities.
3. Solving word problems involving rational equations or inequalities.

To solve a rational equation:

(a) Eliminate denominators by multiplying each term of the equation by the least common denominator.

(b) Note that eliminating denominators may introduce extraneous solutions. Check the solutions of the transformed equations with the original equation.

Example 1. Solve for *x*: $\frac{2}{x} - \frac{3}{2x} = \frac{1}{5}$

Solution. The LCD of all the denominators is 10x. Multiply both sides of the equation by 10x and solve the resolving equation.

$$10x\left(\frac{2}{x}\right) - 10x\left(\frac{3}{2x}\right) = 10x\left(\frac{1}{5}\right)$$

$$20 - 15 = 2x$$

$$5 = 2x$$

$$\frac{5}{2} = x$$

Example 2. Solve for x: $\frac{x}{x+2} - \frac{1}{x-2} = \frac{8}{x^2-4}$

Solution. Factor each denominator in the rational expression.

$$\frac{x}{x+2} - \frac{1}{x-2} = \frac{8}{(x+2)(x-2)}$$

Multiply the LCD to both sides of the equation to remove the denominators.

$$[(x+2)(x-2)]\left(\frac{x}{x+2}\right) - \left(\frac{1}{(x-2)}\right) = [(x+2)(x-2)]\left(\frac{8}{(x+2)(x-2)}\right)$$

$$x(x-2) - (x+2) = 8$$

$$x^2 - 3x - 10 = 0$$

Upon reaching this step, we can use strategies for solving polynomial equations.

$$x^2 - 3x - 10 = 0 \quad \rightarrow \quad (x+2)(x-5) = 0$$

$$x + 2 = 0 \quad \text{or} \quad x - 5 = 0$$

$$x = -2 \quad \text{or} \quad x = 5$$

Since $x = -2$ makes the original equat ion undefined, $x = 5$ is the only solution.

Example 3. In an inter-barangay basketball league, the team from Barangay Culiat has won 12 out of 25 games, a winning percentage of 48%. How many games should they win in a row to improve their win percentage to 60%

Solution. Let x represent the number of games that they need to win to raise their percentage to 60%. The team has already won 12 out of their 25 games. If they win x games in a row to increase their percentage to 60%, then they would have played 12+x games out of their 25+x games. The equation is $\frac{12+x}{25+x} = 0.6$. Multiply 25+x to both sides of the equation and solve the resulting equation.

$$\frac{12+x}{25+x} = 0.6$$

$$12 + x = (25 + x)(0.6)$$

$$12 + x = 0.6(25) + 0.6(x)$$

$$x - 0.6x = 15 - 12$$

$$0.4x = 3$$

$$x = 7.5$$

Therefore, Barangay Culiat needs to win 8 games in a row to raise their winning percentage to 60%.

Example 4. Jens walks 5 kilometers from his house to Quiapo to buy a new bike which he uses to return home. He averaged 10 kilometers faster on his bike than on foot. If his total trip took 1 hour and 20 minutes, what is his walking speed in kph? Use the formula $v = \frac{d}{t}$.

Solution. Using the formula $v = \frac{d}{t}$, we derive the formula for the time $t = \frac{d}{v}$. Let v be Jens' walking speed. Then v+10 is his speed on his new bike. Jens' walking time is $\frac{5}{v}$ and his biking time is $\frac{5}{v+10}$.
The equation now becomes $\frac{5}{v} + \frac{5}{v+10} = \frac{4}{3}$.
Multiply both sides of the equation by the LCD and solve the resulting equation.

$$(3v(v+10))\left(\frac{5}{v} + \frac{5}{v+10}\right) = \left(\frac{4}{3}\right)(3v(v+10))$$

$$15(v+10) + 15v = 4v(v+10)$$

$$30v + 150 = 4v^2 + 40v$$

$$4v^2 + 10v - 150 = 0$$

$$2v^2 + 5v - 75 = 0$$

$$(2v+15)(v-5) = 0$$

$$v = -\frac{15}{2} \; or \; v = 5$$

Rejecting the value $v = -\frac{15}{2}$, we conclude that Jens' walking speed is 5kph.

To solve rational inequalities:

(a) Rewrite the inequality as a single rational expression on one side of the inequality symbol and 0 on the other side.
(b) Determine over what intervals the rational expression takes on positive and negative values.
   i. Locate the x values for which the rational expression is zero or undefined (factoring the numerator and denominator is a useful strategy).
   ii. Mark the numbers found in (i) on a number line. Use a shaded circle to indicate that the value is included in the solution set, and a hollow circle to indicate that the value is excluded. These numbers partition the number line into intervals.
   iii. Select a test point within the interior of each interval in (ii). The sign of the rational expression at this test point is also the sign of the rational expression at each interior point in the aforementioned interval.
   iv. Summarize the intervals containing the solutions.

Warning! Multiplying both sides of an inequality by a number requires that the sign (positive or negative) of the number is known. Since the sign of a variable is unknown, it is <u>not</u> valid to multiply both sides of an inequality by a variable.

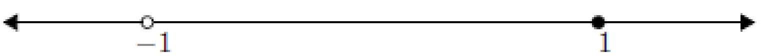
Example 6. Solve the inequality $\frac{2x}{x+1} \geq 1$.

Solution.

(a) Rewrite the inequality as a single rational expression.

$$\frac{2x}{x+1} - 1 \geq 0$$
$$\frac{2x-(x+1)}{x+1} \geq 0$$
$$\frac{x-1}{x+1} \geq 0$$

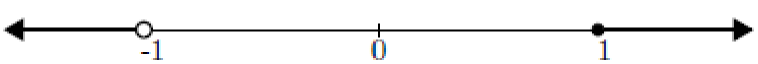
(b) The rational expression will be zero for x = 1 and undefined for x = –1. The value x = 1 is included while x = –1 is not. Mark these on the number line. Use a shaded circle for x = 1 (a solution) and an unshaded circle for x = -1 (not a solution).



(c) Choose convenient test points in the intervals determined by –1 and 1 to determine the sign of $\frac{x-1}{x+1}$ in these intervals. Construct a table of signs as shown below.

Interval	$x < -1$	$-1 < x < 1$	$x > 1$
Test Point	$x = -2$	$x = 0$	$x = 2$
$x - 1$	$-$	$-$	$+$
$x + 1$	$-$	$+$	$+$
$\frac{x-1}{x+1}$	**+**	**–**	**+**

(d) Since we are looking for the intervals where the rational expression is positive or zero, we determine the solution to be the set $\{x \in \mathbb{R} | x < -1 \text{ or } x \geq 1\}$. Plot this set on the number line.



Example 7. Solve the inequality $\frac{3}{x-2} < \frac{1}{x}$.

(a) Rewrite the inequality with zero on one side.

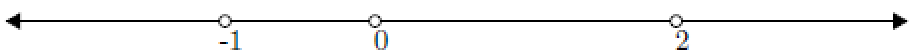
$$\frac{3}{x-2} - \frac{1}{x} < 0$$

$$\frac{3x-(x-2)}{x(x-2)} < 0$$

$$\frac{2x+2}{x(x-2)} < 0$$

$$\frac{2(x+1)}{x(x-2)} < 0$$

(b) The rational expression will be zero for x = –1 and undefined for 0 and 2. Plot these points on a number line. Use hollow circles since these values are not part of the solution.



(c) Construct a table of signs to determine the sign of the function in each interval determined by –1, 0, and 2.

Interval	$x < -1$	$-1 < x < 0$	$0 < x < 2$	$x > 2$
Test Point	$x = -2$	$x = -\frac{1}{2}$	$x = 1$	$x = 3$
$2(x+1)$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$\frac{2(x+1)}{x(x-2)}$	**–**	**+**	**–**	**+**

(d) Summarize the intervals satisfying the inequality. The solution set of the inequality is the set $\{x \in \mathbb{R} | x < -1 \text{ or } 0 < x < 2\}$. Plot this set on the number line.

Example 8. A box with a square base is to have a volume of 8 cubic meters. Let x be the length of the side of the square base and h be the height of the box. What are the possible measurements of a side of the square base if the height should be longer than a side of the square base?

Solution. The volume of a rectangular box is the product of its width, length, and height. Since the base of the box is square, its width and length are equal.

The variable x is the length of a side of the box, while h is its height. The equation relating h and x is $8 = x^2h$. Expressing h in terms of x, we obtain

$$h = \frac{8}{x^2}$$

Since the height is greater than the width, $h > x$ and our inequality is $\frac{8}{x^2} > x$.

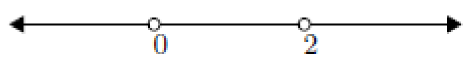
(a) To solve this inequality, we begin by rewriting it with zero on one side:

$$\frac{8}{x^2} - x > 0$$

$$\frac{8 - x^3}{x^2} > 0$$

$$\frac{(2 - x)(x^2 + 2x + 4)}{x^2} > 0$$

(b) The rational expression will be zero for x = 2 and undefined for x = 0. Plot on a number line and use hollow circles since these values are not part of the solution.



(c) Construct a table of signs to determine the sign of the function in each interval determined by 0 and 2. Note that $x^2 + 2x + 4$ is positive for any real values of x.

Interval	$x < 0$	$0 < x < 2$	$x > 2$
Test point	$x = -1$	$x = 1$	$x = 3$
$2 - x$	$+$	$+$	$-$
$x^2 + 2x + 4$	$+$	$+$	$+$
x^2	$+$	$+$	$+$
$\frac{(2 - x)(x^2 + 2x + 4)}{x^2}$	**+**	**+**	**−**

(d) Since the rational expression is positive in the interval 0 < x < 2, this is the solution set of the inequality. We reject the interval x < 0 even if the expression is positive here since we only consider positive values of x for this problem.

Therefore the height of the box should be less than 2 meters.

Example 9. A dressmaker ordered several meters of red cloth from a vendor, but the vendor only had 4 meters of red cloth in stock. The vendor bought the remaining lengths of red cloth from a wholesaler for P1,120.00. He then sold those lengths of red cloth to the dressmaker along with the original 4 meters of cloth for a total of P1,600.00. If the vendor's price per meter is at least P10.00 more than the wholesaler's price per meter, how many additional meters of red cloth did the vendor purchase from the wholesaler?

Solution. Let the variable x be the length of the additional cloth purchased by the vendor from the wholesaler.

The wholesaler's price of red cloth per meter can be expressed as $\frac{1120}{x}$. The vendor's price of red cloth per meter can be expressed as $\frac{1600}{x+4}$.

If the vendor sold his cloth to the dressmaker at a price that is at least P10.00 more per meter, the inequality can be written as

$$\frac{1600}{x+4} \geq \frac{1120}{x} + 10$$

(a) To solve this inequality, we rewrite with zero on one side and simplify:

$$\frac{1600}{x+4} \geq \frac{1120}{x} + 10$$

$$\frac{1600}{x+4} - \frac{1120}{x} \geq 10$$

$$\frac{1600}{x+4} - \frac{1120}{x} \geq 10$$

$$\frac{160}{x+4} - \frac{112}{x} \geq 1$$

$$\frac{160}{x+4} - \frac{112}{x} - 1 \geq 0$$

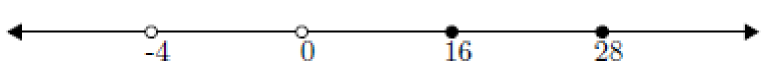
$$\frac{160x - 112(x+4) - (x^2 + 4x)}{x(x+4)} \geq 0$$

$$\frac{160x - 112x - 448 - x^2 - 4x}{x(x+4)} \geq 0$$

$$\frac{x^2 - 44x + 448}{x(x+4)} \leq 0$$

$$\frac{(x-16)(x-28)}{x(x+4)} \leq 0$$

(b) The rational expression will be zero for x = 16 and x = 28 and undefined for x = 0 and x = 4. Plot on a number line and use hollow circles since these values are not part of the solution set. The figure below is not drawn to scale.



(c) Construct a table of signs to determine the sign of the function in each interval determined by the values above.

Interval	$x < -4$	$-4 < x < 0$	$0 < x < 16$	$16 < x < 28$	$x > 28$
Test point	$x = -5$	$x = -1$	$x = 10$	$x = 20$	$x = 30$
$x - 16$	$-$	$-$	$-$	$+$	$+$
$x - 28$	$-$	$-$	$-$	$-$	$+$
x	$-$	$-$	$+$	$+$	$+$
$x + 4$	$-$	$+$	$+$	$+$	$+$
$\frac{(x-16)(x-28)}{x(x+4)}$	**+**	**–**	**+**	**–**	**+**

(d) The rational expression is negative in the interval -4 < x < 0 and in the interval 16 < x < 28. However, since we are dealing with lengths of cloth, we discard the interval where the length is negative. Also, the rational expression is 0 when x = 16 and x = 28. **Therefore the vendor bought and sold an additional length of red cloth from 16 – 28 meters to the dressmaker**.

Solved Examples

1. Solve for x: $\frac{x+6}{x-4} = \frac{1}{x+1}$.

Solution.
The LCD is $(x-4)(x+1)$. Multiplying both sides of the equation by the LCD results in:

$$(x+6)(x-4) = (x-4)$$

$$x^2 + 2x - 24 = x - 4$$

$$x^2 + x - 20 = 0$$

$$(x+5)(x-4) = 0$$

Therefore, *x* = –5 or *x* = 4. Since *x* = *4* will make the original equation undefined, the only solution is *x* = –5*.*

2. Solve for x: $\frac{5x}{x-1} < 4$.

***Solution*.**
(a) Rewrite the equation with zero on one side

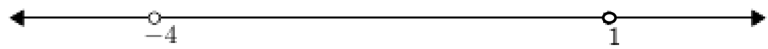
$$\frac{5x}{x-1} - 4 < 0$$

$$\frac{5x}{x-1} - \frac{4(x-1)}{x-1} < 0$$

$$\frac{5x - 4x + 4}{x-1} < 0$$

$$\frac{x+4}{x-1} < 0$$

(b) The rational expression will be zero for x = -4 and undefined for x = 1. Plot the points on a number line and use hollow circles since these values are not part of the solution set.



(c) Construct a table of signs to determine the sign of the function in each interval determined by the values above.

Interval	$x < -4$	$-4 < x < 1$	$x > 1$
Test Point	$x = -5$	$x = 0$	$x = 2$
$x + 4$	$-$	$+$	$+$
$x - 1$	$-$	$-$	$+$
$\dfrac{x+4}{x-1}$	**+**	**−**	**+**

(d) We are looking for the intervals where the function will be negative. The solution set is given by $\{x \in \mathbb{R} \mid -4 < x < 1\}$. The graph is shown below.

3. Solve for x: $\frac{4}{2x-1} \geq \frac{1}{x+1}$

Solution.

(a) Rewrite the equation with zero on one side

$$\frac{4}{2x-1} - \frac{1}{x+1} \geq 0$$

$$\frac{4(x+1)-(2x-1)}{(2x-1)(x+1)} \geq 0$$

$$\frac{4x+4-2x+1}{(2x-1)(x+1)} \geq 0$$

$$\frac{2x+5}{(2x-1)(x+1)} \geq 0$$

(b) The rational expression will be zero for $x = -\frac{5}{2}$ and undefined for $x = \frac{1}{2}$ and $x = -1$. Mark these on the number line where $x = -\frac{5}{2}$ is included while the others are not.

(c) Construct a table of signs to determine the sign of the function in each interval determined by the values above.

Interval	$x < -\frac{5}{2}$	$-\frac{5}{2} < x < -1$	$-1 < x < \frac{1}{2}$	$x > \frac{1}{2}$
Test Point	$x = -3$	$x = -2$	$x = 0$	$x = 1$
$2x + 5$	$-$	$+$	$+$	$+$
$x + 1$	$-$	$-$	$+$	$+$
$2x - 1$	$-$	$-$	$-$	$+$
$\frac{2x+5}{(2x-1)(x+1)}$	**−**	**+**	**−**	**+**

(d) The solution set of the inequality is given by $\{x \in | -\frac{5}{2} \leq x < -1 \text{ } or \text{ } x > \frac{1}{2}\}$

Lesson 6 Supplementary Exercises

1. Solve for x: $\frac{2x-1}{x+3} = 5$

2. Solve for x: $\frac{x^2}{x-3} = \frac{x+2}{2x-5}$

3. Solve for x: $\frac{x-1}{x+3} > 0$

4. Solve for x: $\frac{1}{x} < \frac{1}{x-3}$

5. Solve for x: $\frac{x^2-x-30}{x-1} \geq 0$

6. If a and b are real numbers such that $a < b$, find the solution set of $\frac{x-a}{b-x} \leq 0$.

7. You have 6 liters of a pineapple juice blend that has 50% pure pineapple juice. How many liters of pure pineapple juice needs to be added to make a juice blend that is 75% pineapple juice?

8. Two ships traveling from Dumaguete to Cagayan de Oro differ in average speed by 10 kph. The slower ship takes 3 hours longer to travel a 240-kilometer route than for the faster ship to travel a 200-kilometer route. Find the speed of the slower ship.

Lesson 7: Representations of Rational Functions

Learning Outcome(s): At the end of the lesson, the learner is able to represent a rational function through its table of values, graphs and equation, and solve problems involving rational functions.

Lesson Outline:

1. Table of values, graphs and equations as representations of a rational function.
2. Rational functions as representations of real-life situations

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Definition: A **rational function** is a function of the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$and $q(x)$ are polynomial functions and $q(x)$ is not the zero polynomial (i.e., $q(x) \not\equiv 0$). The domain of $f(x)$ is all values of x where $q(x) \neq 0$.

Average speed (or velocity) can be computed by the formula $s = d/t$. Consider a 100-meter track used for foot races. The speed of a runner can be computed by taking the time for him to run the track and applying it to the formula $s = \frac{100}{t}$, since the distance is fixed at 100 meters.

Example 1. Represent the speed of a runner as a function of the time it takes to run 100 meters in the track.

Solution. Since the speed of a runner depends on the time it takes to run 100 meters, we can represent speed as a function of time.

Let x represent the time it takes to run 100 meters. Then the speed can be represented as a function $s(x)$ as follows:

$$s(x) = \frac{100}{x}$$

Observe that it is similar to the structure to the formula $s = \frac{d}{t}$ relating speed, distance, and time.

Example 2. Continuing the scenario above, construct a table of values for the speed of a runner against different run times.

Solution. A table of values can help us determine the behavior of a function as the variable x changes.

The current world record (as of October 2015) for the 100-meter dash is 9.58 seconds set by the Jamaican Usain Bolt in 2009. We start our table of values at 10 seconds.

Let x be the runtime and $s(x)$ be the speed of the runner in meters per second, where $s(x) = \frac{100}{x}$. The table of values for run times from 10 seconds to 20 seconds is as follows:

x	10	12	14	16	18	20
$s(x)$	10	8.33	7.14	6.25	5.56	5

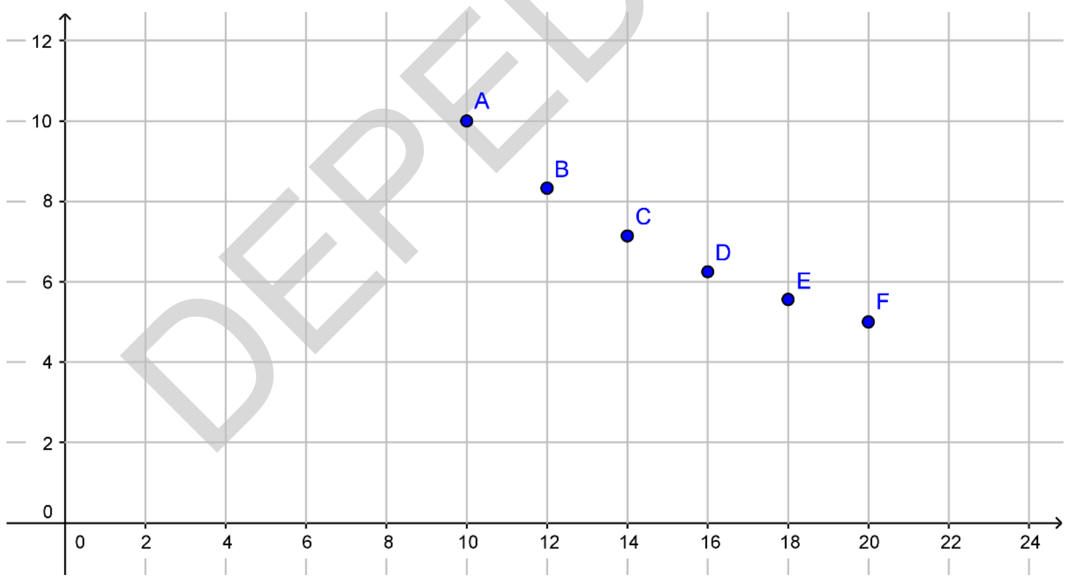
From the table we can observe that the speed decreases with time. We can use a graph to determine if the points on the function follow a smooth curve or a straight line.

Example 3. Plot the points on the table of values on a Cartesian plane. Determine if the points on the function $s(x) = \frac{100}{x}$ follow a smooth curve or a straight line.

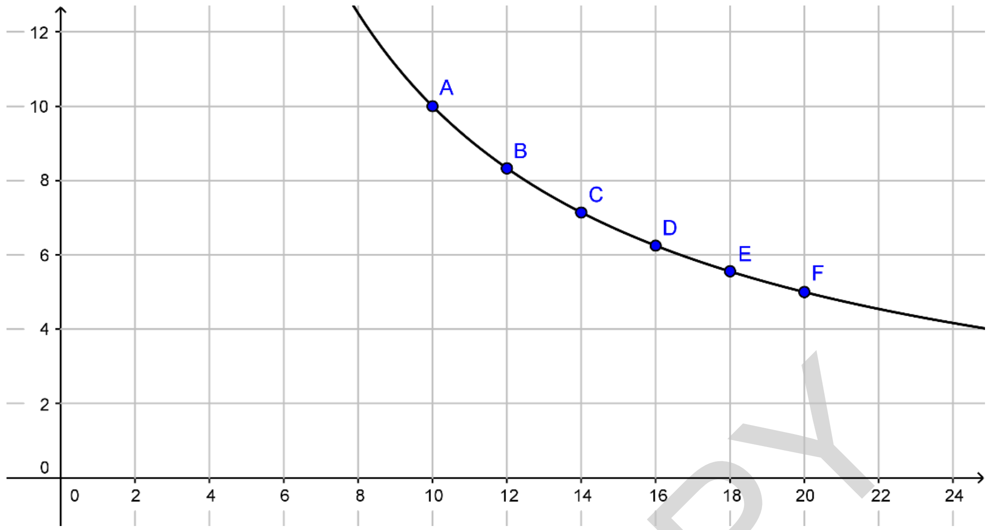
Solution. Assign points on the Cartesian plane for each entry on the table of values above:

A(10,10) B(12,8.33) C(14, 7.14) D(16, 6.25) E(18,5.56) F(20,5)

Plot these points on the Cartesian plane:



By connecting the points, we can see that they are not collinear but rather follows a smooth curve.



For the 100-meter dash scenario, we have constructed a function of speed against time, and represented our function with a table of values and a graph.

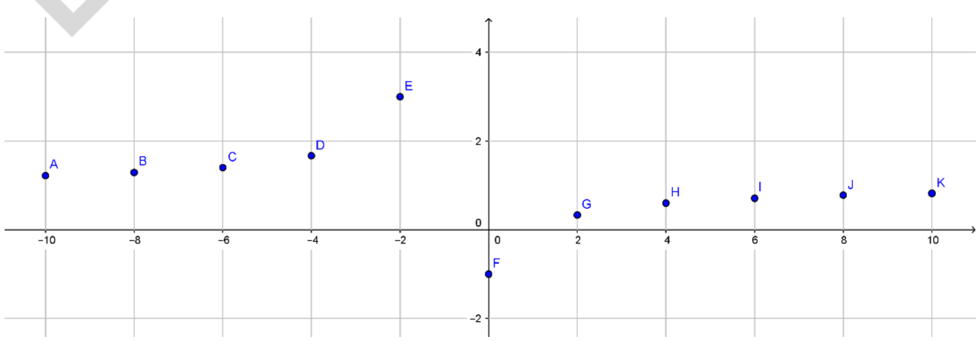
The previous example is based on a real world scenario and has limitations on the values of the *x*-variable. For example, a runner cannot have negative time (which would mean he is running backwards in time!), nor can he exceed the limits of human physiology (can a person run 100-meters in 5 seconds?). However, we can apply the skills of constructing tables of values and plotting graphs to observe the behavior of rational functions.

Example 4. Represent the rational function given by $f(x) = \frac{x-1}{x+1}$ using a table of values and plot a graph of the function by connecting points.

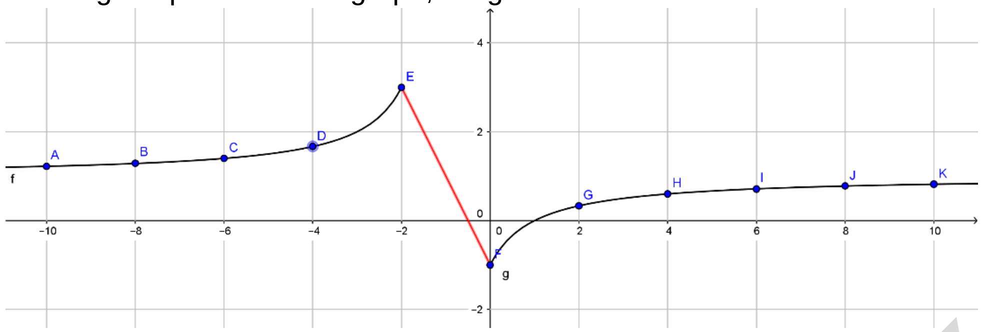
Solution. Since we are now considering functions in general, we can find function values across more values of *x*. Let us construct a table of values for some *x*-values from -10 to 10:

x	-10	-8	-6	-4	-2	0	2	4	6	8	10
$f(x)$	1.22	1.29	1.4	1.67	3	-1	0.33	0.6	0.71	0.78	0.82

Plotting the points on a Cartesian plane we get:



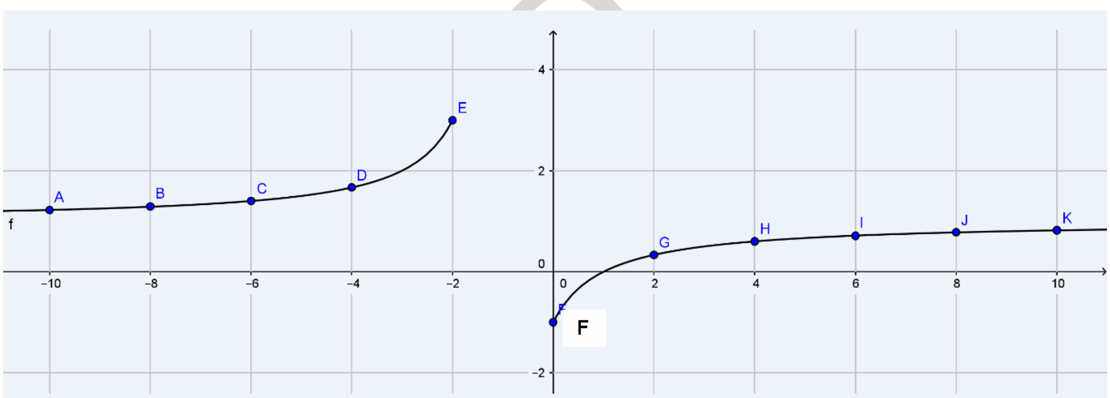
Connecting the points on this graph, we get:



Why would the graph unexpectedly break the smooth curve and jump from point E to point F? The answer is that it doesn't! Let us take a look at the function again:

$$f(x) = \frac{x-1}{x+1}$$

Observe that the function will be undefined at $x = -1$. This means that there cannot be a line connecting point E and point F as this implies that there is a point in the graph of the function where $x = -1$.We will cover this aspect of graphs of rational functions in a future lesson, so for now we just present a partial graph for the function above as follows:

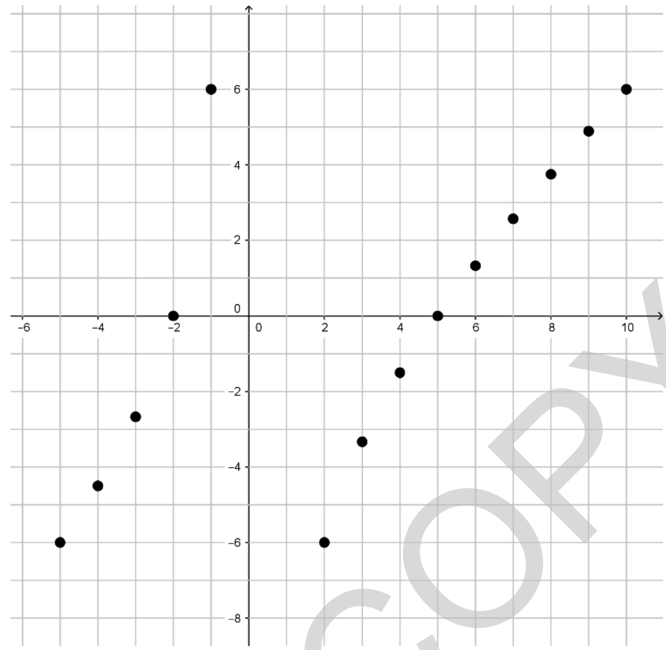


Example 5. Represent the rational function $f(x) = \frac{x^2-3x-10}{x}$ using a table of values. Plot the points given in the table of values and sketch a graph by connecting the points.

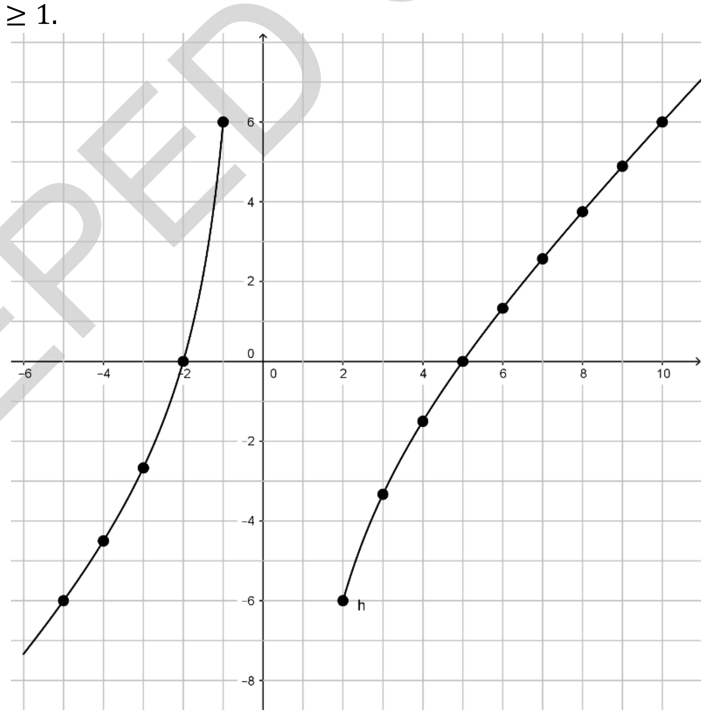
Solution. As we have seen in the previous example, we will need to take a look at the *x*-values which will make the denominator zero. In this function, $x = 0$ will make the denominator zero. Taking function values for integers in$-6 \leq x \leq 10, x \neq 0$ we get the following table of values:

x	-5	-4	-3	-2	-1	1	2	3	4	5	6	7	8	9	10
$f(x)$	-6	-4.5	-2.67	0	6	-12	-6	-3.33	-1.5	0	1.33	2.57	3.75	4.89	6

Plotting the values above as points in the Cartesian plane:



We connect the dots to sketch the graph, but we keep in mind that $x = 0$ is not part of the domain. For now we only connect those with values $x \leq -1$ and those with values $x \geq 1$.



Note that $x = -2$ and $x = 5$ are zeroes of the rational function, which means that the function value at these values is zero. These *x*-values give the *x*-intercepts of the graph.

The behavior of the function near those values which make the function undefined will be studied in the next few lessons.

Example 6. In an inter-barangay basketball league, the team from Barangay Culiat has won 12 out of 25 games, a winning percentage of 48%. We have seen that they need to win 89 games consecutively to raise their percentage to at least 60%. What will be their winning percentage if they win:

(a) 10 games in a row?
(b) 15? 20? 30? 50? 100 games?
(c) Can they reach a 100% winning percentage?

Solution. Let x be the number of wins the Barangay Culiat needs to win in a row. Then the percentage p is a function of the number of wins that the team needs to win. The function can be written as:

$$p(x) = \frac{12 + x}{25 + x}$$

Construct a table of values for $p(x)$:

x	10	15	20	30	50	100	200	300
$p(x)$	0.63	0.68	0.71	0.76	0.83	0.90	0.94	0.96

We interpret the table of values as follows:

No. of consecutive wins	Win Percentage
8	60%
10	63%
15	68%
20	71%
30	76%
50	83%
100	90%
200	94%
300	96%

Even if the team wins 300 consecutive games, the team still cannot reach a 100% winning percentage. Note that the denominator 25 + x is greater than the numerator, so 100% cannot be achieved. This is reasonable, since the team has lost 13 games already, so they cannot be "perfect".

Example 7. Ten goats were set loose in an island and their population growth can be approximated by the function $P(t) = \left\lfloor \frac{60(t+1)}{t+6} \right\rfloor$

where *P* represents the goat population in year *t* since they were set loose. Recall that the symbol $\lfloor . \rfloor$denotes the greatest integer function.

(a) How many goats will there be after 5 years?
(b) What is the maximum goat population that the island can support?

Solution.

(a) Evaluate the function for $t = 5$:

$$P(5) = \left\lfloor \frac{60(5+1)}{5+6} \right\rfloor = \lfloor 32.726 \rfloor = 32$$

There will be 32 goats after 5 years.

(b) Construct a table of values for $P(x)$:

t	5	10	15	20	50	300	1000
$P(t)$	32	41	45	48	54	59	59

Observe that even if t increase, the function does not exceed 59. The model suggests that the island can only support up to 59 goats. (Note that since the model is just an approximation, there may be errors and the number 59 may not be exact).

Solved Examples

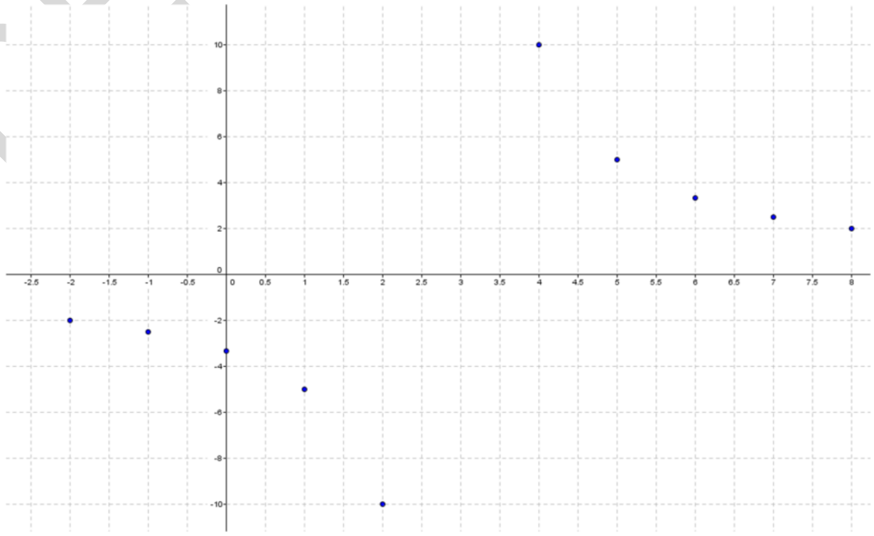
1. Given $f(x) = \frac{10}{x-3}$,

(a) Construct a table of values using the numbers from -2 to 8.
(b) Plot the points in the Cartesian plane and determine whether the points form a smooth curve or a straight line.

Solution.
(a)

x	-2	-1	0	1	2	3	4	5	6	7	8
$f(x)$	-2	-2.5	-3.3	-5	-10	Und.	10	5	3.3	2.5	2

(b) Plotting the points, we get the following:



Connecting the points, we get the following graph which forms two different smooth curves.

2. Using integer values from -5 to 5, find the interval where the smooth curve of the following functions will disconnect:

(a) $f(x) = \frac{4}{x-1}$

(b) $g(x) = \frac{x-3}{x+1}$

***Solution*.**

(a)

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$	-0.66	-0.8	-1	-1.33	-2	-4	und	4	2	1.33	1

Constructing the table of values, we see that the function $f(x)$is undefined at $x = 1$, If we plot the points and connect them, we will see that we can only connect those with values $x \leq 0$ and those with values $x \geq 2$.

So we can say that it disconnects at the interval $(0,2)$.

(b)

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$g(x)$	2	2.33	3	5	und	-3	-1	-0.33	0	0.8	0.33

Similar to letter (a), we see that $g(x)$ is undefined at $x = -1$. It disconnects at the interval $(-2,0)$.

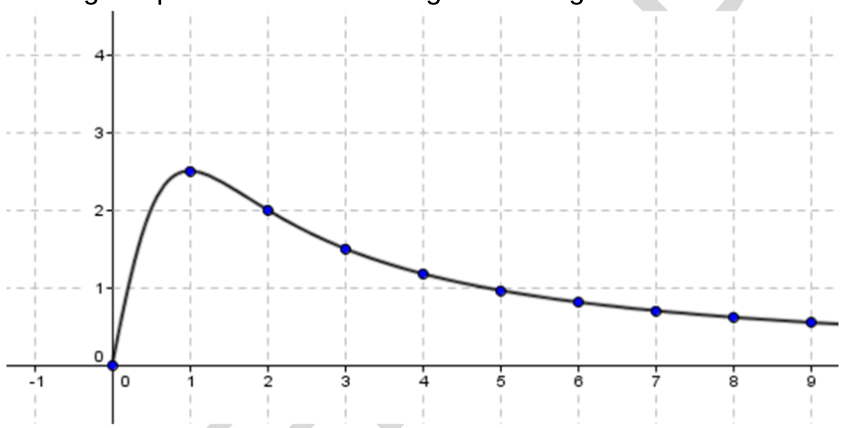
3. A hypothetical function representing the concentration of a drug in a patient's bloodstream over time t (in hours) is given as $c(t) = \frac{5t}{t^2+1}$.
   (a) Construct a table of values.
   (b) Plot the points in a Cartesian plane and connect them.
   (c) What can you say about the function?

***Solution*.**

(a) Since t is in time, we can only use non-negative values for it. Using the first ten whole numbers, we get

t	0	1	2	3	4	5	6	7	8	9
c(t)	0	2.5	2	1.5	1.18	0.96	0.81	0.7	0.62	0.55

(b) Plotting the points and connecting them we get:



(c) At $t = 0$, the concentration is zero since the drug has not entered the bloodstream yet. It shoots up at $t = 1$ but it starts decreasing after that.

Lesson 7 Supplementary Exercises

1. Construct a table of values for the following functions using the integers from -4 to 4.
   (a) $f(x) = \frac{6}{x-2}$ (b) $r(x) = \frac{6x+12}{x^2-4}$

2. Using the table of values you got from the previous question, plot and connect the points of
   (c) $f(x)$ (e) $r(x)$

3. A certain invasive species of fish was introduced in a small lake and their population growth can be modeled with time t by the function

$$f(t) = \left\lfloor \frac{1000(t+1)}{t+10} \right\rfloor$$

   (a) Construct a table of values
   (b) Is their population approaching a specific value?

Lesson 8: Graphing Rational Functions

Learning Outcome(s): At the end of the lesson, the learner is able to find the domain and range, intercepts, zeroes, asymptotes of rational functions, graph rational functions, and solve problems involving rational functions.

Lesson Outline:

1. Domain and range of rational functions.
2. Intercepts and zeroes of rational functions.
3. Vertical and horizontal asymptotes of rational functions.
4. Graphs of rational functions

Recall:

(a) The **domain** of a function is the set of all values that the variable x can take.
(b) The **range** of the function is the set of all values that f(x) will take.
(c) The **zeroes** of a function are the values of x which make the function zero. The real numbered zeroes are also **x-intercepts** of the graph of the function.
(d) The **y-intercept** is the function value when x=0.

Example 1. Consider the function $f(x) = \frac{x-2}{x+2}$. (a) Find its domain, (b) intercepts, (c) sketch its graph and (d) determine its range.

Solution.

(a) The domain of f(x) is $\{x \in \mathbb{R} | x \neq -2\}$.
Observe that the function is undefined at x = –2. This means that x = -2 is not part of the domain of f(x). In addition, other values of x will make the function undefined.

(b) The x-intercept of f(x) is 2 and its y-intercept is –1.
Recall that the x-intercepts of a rational function are the values of x that will make the function zero. A rational function will be zero if its numerator is zero. Therefore the zeroes of a rational function are the zeroes of its numerator.
The numerator x – 2 will be zero at x=2. Therefore x=2 is a zero of f(x). Since it is a real zero, it is also an x-intercept.
The y-intercept of a function is equal to f(0). In this case, $f(0) = -\frac{2}{2} = -1$.

(c) In sketching the graph of f(x), let us look at what happens to the graph near the values of x which make the denominator undefined. Recall that in the previous lesson, we simply skipped connecting the points at integer values. Let us see what happens when x takes on values that brings the denominator closer to zero.

The denominator is zero when x = –2. Let us look at the values of x close to –2 on its left side (i.e. x < –2, denoted -2^{-}) and values of x close to –2 on its right side (i.e. x > –2, denoted -2^{+}).

i. Table of values for x approaching -2^{-}

x	–3	–2.5	–2.1	–2.01	–2.001	–2.0001	As *x* approaches –2
$f(x)$	5	9	41	401	4001	40001	*f(x)* increases without bound.

Notation. We use the notation "$f(x) \to +\infty$ as $x \to -2^{-}$" to indicate that f(x) increases without bound as x approaches –2 from the left.

ii. Table of values for x approaching -2^{+}

x	–1	–1.5	–1.9	–1.99	–1.999	–1.9999	As *x* approaches -2^{+}
$f(x)$	–3	–7	–39	–399	–3999	–39999	*f(x)* decreases without bound.

Notation. We use the notation "$f(x) \to -\infty$ as $x \to -2^{+}$" to indicate that f(x) decreases without bound as x approaches –2 from the right.

Plotting the points corresponding to these values on the Cartesian plane:

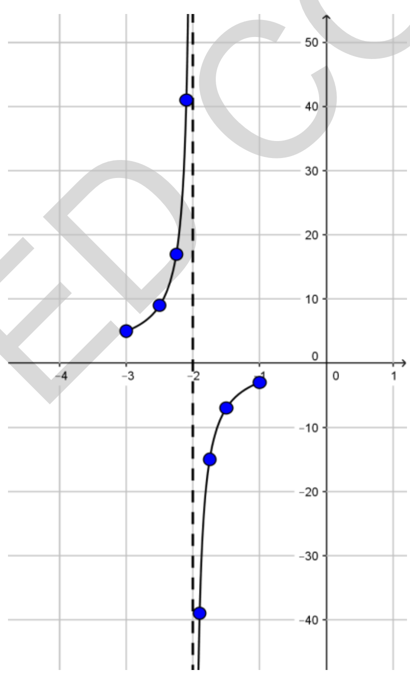


Figure 2.3 Note that the axes do not have the same scale.

Observe that as x approaches –2 from the left and from the right, f(x) gets closer and closer to the line x = –2, indicated in the figure with a dashed line.

We call this line a vertical asymptote, formally defined as follows:

Definition. The vertical line x = a is a **vertical asymptote** of a function f if the graph of f either increases or decreases without bound as the x-values approach a from the right or left.

Finding the Vertical Asymptotes of a Rational Function

- Find the values of *a* where the denominator is zero.
- If this value of *a* does not make the numerator zero, then the line x = a is a vertical asymptote.

We will also look how the function behaves as x increases or decreases without bound.

We first construct a table of values for f(x) as x increases without bound, or in symbols, as $x \to +\infty$.

iii. Table of values for f(x) as $x \to +\infty$

x	5	10	100	1,000	10,000	As $x \to +\infty$
$f(x)$	0.43	0.67	0.96	0.9960	0.99960	f(x) approaches 1^-

Next, construct a table of values for f(x) as x decreases without bound, or in symbols, as $x \to -\infty$.

iv. Table of values for f(x) as $x \to -\infty$

x	–5	–10	–100	–1,000	–10,000	As $x \to -\infty$
$f(x)$	2.33	1.41	1.041	1.00401	1.0004001	f(x) approaches 1^+

Plotting the points according to these on the Cartesian Plane:

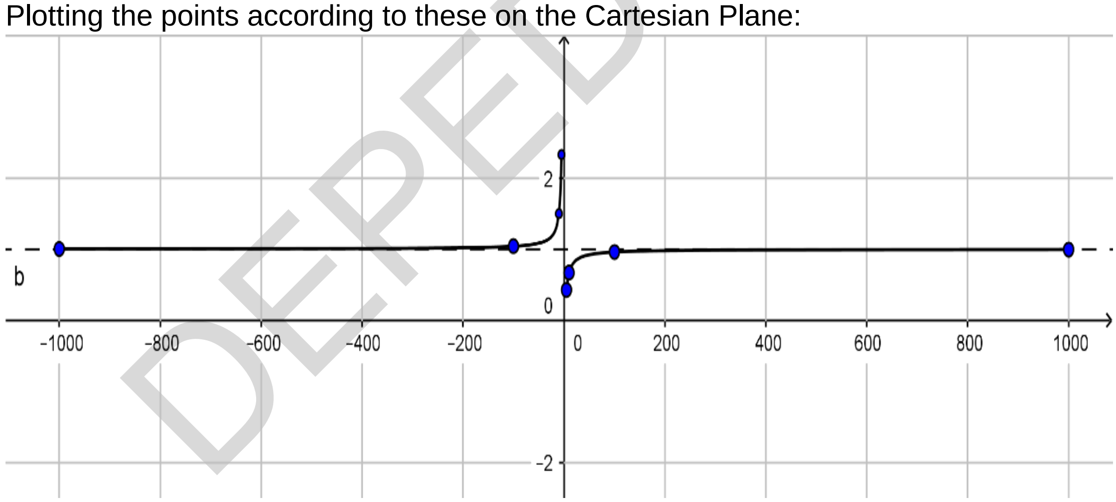


Figure 2.4: Note that the axes do not have the same scale.

Observe that as x increases or decreases without bound, f(x) gets closer and closer to 1. The line y= 1 is indicated in the figure with a dashed line.

We call this line a horizontal asymptote, formally defined as follows:

Definition. The horizontal line y = b is a **horizontal asymptote** of the function f if f(x) gets closer to b as x increases or decreases without bound ($x \to +\infty$ or $x \to -\infty$.).

A rational function may or may not cross its horizontal asymptote. If the function does not cross the horizontal asymptote y = b, then b is not part of the range of the rational function.

Now that we know the behavior of the function as x approaches –2 (where the function is not defined), and also as $x \to +\infty$ or $x \to -\infty$, we can complete the sketch of the graph by looking at the behavior of the graph at the zeroes.

Construct a table of signs to determine the sign of the function on the intervals determined by the zeroes and the intercepts. Refer to the lesson on rational inequalities for the steps in constructing a table of signs:

Interval	$x < -2$	$-2 < x < 2$	$x > 2$
Test point	$x = -3$	$x = 0$	$x = 3$
Test with the rational function	$f(x) > 0$	$f(x) < 0$	$f(x) > 0$

The boundary between the intervals -2 < x < 2 and x >2 is a zero. Since the function is positive on the left of 2 and negative on the right, the function transitions from positive to negative at x = 2.

Plot the zeroes, y-intercept, and the asymptotes. From the table of signs and the previous graphs, we know that f(x)<1 as $x \to -\infty$. Draw a short segment across (2,0) to indicate that the function transitions from negative to positive at this point.

We also know that f(x) increases without bound as $x \to -2^-$ and f(x) decreases without bound as as$x \to -2^+$. Sketch some arrows near the asymptote to indicate this information.

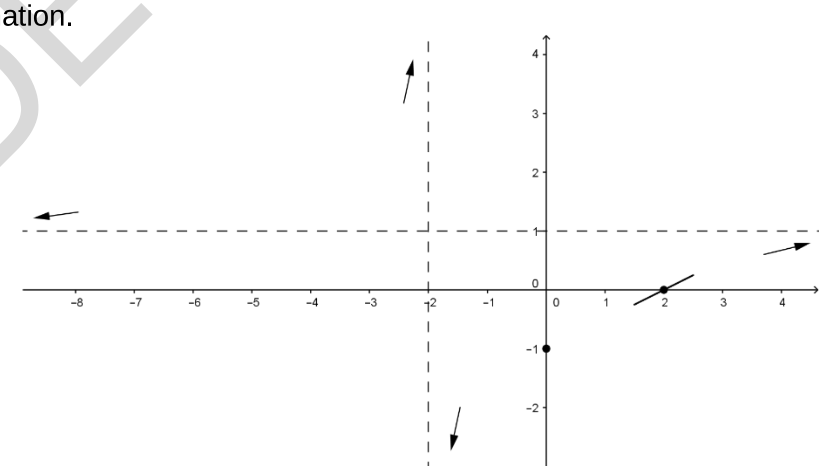


Figure 2.5 Zeroes and asymptotes of f(x).

Trace the arrowheads along with the intercepts using smooth curves. Do not cross the vertical asymptote.

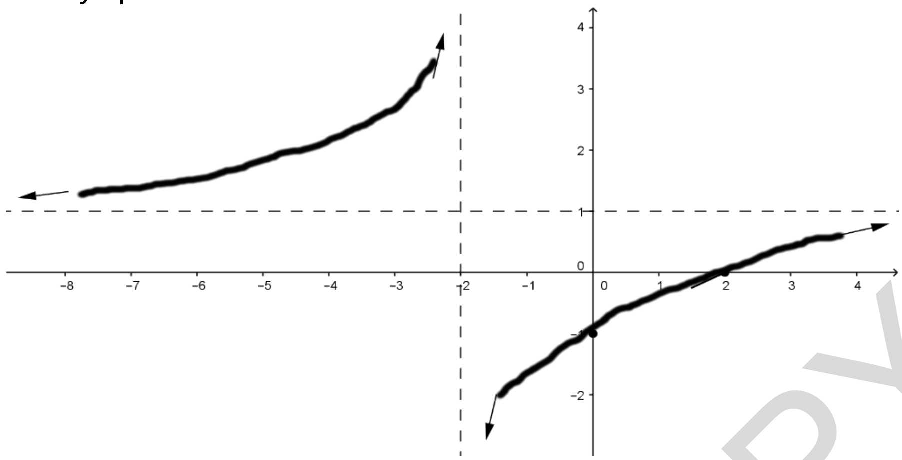


Figure 2.6 Tracing with smooth curves

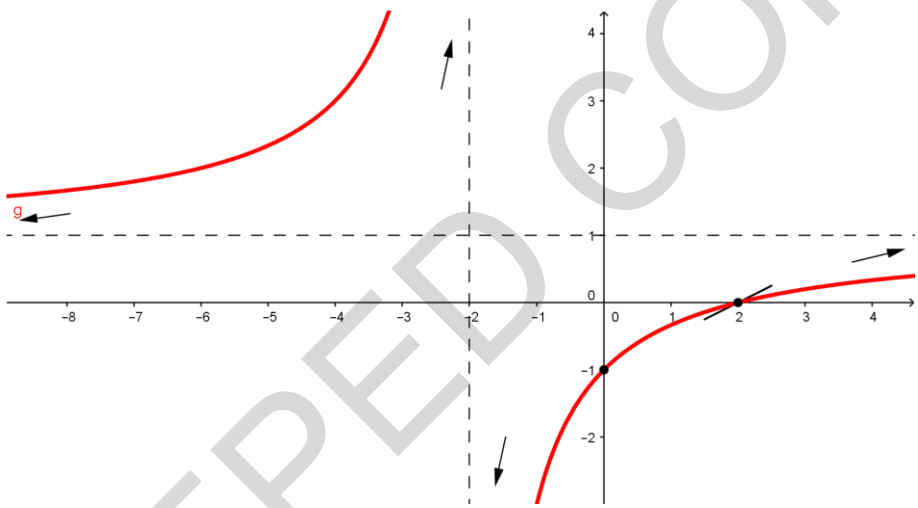


Figure 2.7 The actual sketch of the graph of $y = \frac{x-2}{x+2}$ for reference.

(d) From the graph of the rational function, we observe that the function does not cross the horizontal asymptote. We also observe that the function increases and decreases without bound, and is asymptotic to the line y = 1. Therefore only the value 1 is not included in the range of f(x).

The **range** of f(x) is $\{y \in \mathbb{R} | y \neq 1\}$.

Example 2. Find the horizontal asymptote of $f(x) = \frac{4x^2+4x+1}{x^2+3x+2}$.

Solution. We have seen from the previous example that the horizontal asymptotes can be determined by looking at the behavior of rational functions when |x| is very large (i.e., at extreme values of x).

However, at extreme values of x, the value of a polynomial can be approximated using the value of the leading term.

For example, if x=1000, the value of $4x^2 + 4x + 1$ is 4,004,001. A good approximation is the value of $4x^2$, which is 4,000,000.

Similarly, for extreme values of x, the value of $x^2 + 3x + 2$ can be approximated by x^2. Thus, for extreme values of x, then $f(x)$ can be approximated by $\frac{4x^2}{x^2} = 4$, and therefore $f(x)$ approaches 4 for extreme values of x.

This means that we have a **horizontal asymptote** at y=4.

Example 3. Find the horizontal asymptote of $f(x) = \frac{2x^2-5}{3x^2+x-7}$.

Solution. Following the idea from the previous example, the value of $\frac{2x^2-5}{3x^2+x-7}$ can be approximated by $\frac{2x^2}{3x^2} = \frac{2}{3}$ for extreme values of x.

Thus, the **horizontal asymptote** is $y = \frac{2}{3}$.

Example 4. Find the horizontal asymptote of $f(x) = \frac{3x+4}{2x^2+3x+1}$.

Solution. Again, based on the idea from the previous example, the value of $\frac{3x+4}{2x^2+3x+1}$ can be approximated by $\frac{3x}{2x^2} = \frac{3}{2x}$ for extreme values of x.

If we substitute extreme values of x in $\frac{3}{2x}$, we obtain values very close to 0.

Thus, the **horizontal asymptote** is y=0.

Example 5. Show that $f(x) = \frac{4x^3-1}{3x^2+2x-5}$ can be approximated by $\frac{4x^3}{3x^2} = \frac{4x}{3}$.

If we substitute extreme values of x in $\frac{4x}{3}$, we obtain extreme values as well.

Thus, if x takes on extreme values, then y also takes on extreme values and does not approach a particular finite number. The function has **no horizontal asymptote**.

We summarize the results from the previous examples as follows:

Finding the Horizontal Asymptotes of a Rational Function
Let n be the degree of the numerator and m be the degree of the denominator.

- If $n < m$, the **horizontal asymptote** is $y = 0$.
- If $n = m$, the **horizontal asymptote** is $y = \frac{a}{b}$, where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator.
- If $n > m$, there is no **horizontal asymptote**.

Properties of rational functions:

How to find the:	**Do the following:**
y-intercept	Evaluate the function at x = 0.
x-intercept	Find the values of x where the numerator will be zero.
Vertical asymptotes	Find the values of *a* where the denominator is zero. If this value of *a* does not make the numerator zero, then the line x = a is a vertical asymptote.
Horizontal asymptotes	Let n be the degree of the numerator and m the degree of the denominator • If n < m, the horizontal asymptote is y = 0. • If n = m, the horizontal asymptote is $y = \frac{a}{b}$, where **a** is the leading coefficient of the numerator and **b** is the leading coefficient of the denominator. • If n > m, there is no horizontal asymptote.

Example 6. Sketch the graph of $(x) = \frac{3x^2-8x-3}{2x^2+7x-4}$. Find its domain and range.

Solution. The numerator and denominator of f(x) can be factored as follows:

$$f(x) = \frac{3x^2 - 8x - 3}{2x^2 + 7x - 4} = \frac{(3x + 1)(x - 3)}{(2x - 1)(x + 4)}$$

From the factorization, we can get the following properties of the function:

- y-intercept: $f(0) = \frac{0-0-3}{0+0-4} = \frac{3}{4}$
- zeroes: $\left\{-\frac{1}{3}, 3\right\}$
- vertical asymptotes: $x = \frac{1}{2}$ and $x = -4$
- horizontal asymptote: The polynomials in the numerator and denominator have equal degree. The horizontal asymptote is the ratio of the leading coefficients: $y = \frac{3}{2}$

Plot the intercepts and asymptotes on the Cartesian plane:

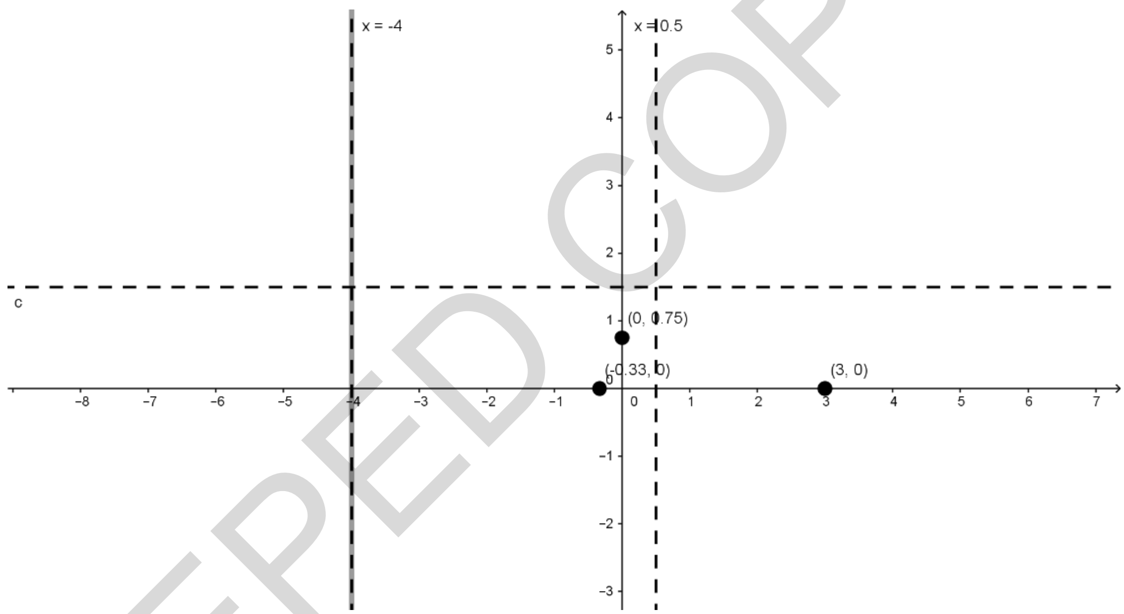


Figure 2.8: Intercepts and asymptotes of f(x).

Construct a table of signs for the following intervals defined by the zeroes and the values where the denominator will be zero:

- $x < -4$
- $-4 < x < -\frac{1}{3}$
- $-\frac{1}{3} < x < \frac{1}{2}$
- $\frac{1}{2} < x < 3$
- $x > 3$

Interval	$x < -4$	$-4 < x < -\frac{1}{3}$	$-\frac{1}{3} < x < \frac{1}{2}$	$\frac{1}{2} < x < 3$	$x > 3$
Test point	$x = -10$	$x = -2$	$x = 0$	$x = 1$	$x = 10$
3x + 1	–	–	+	+	+
x – 3	–	–	–	–	+
2x – 1	–	–	–	+	+
x + 4	–	+	+	+	+
$\frac{(3x+1)(x-3)}{(2x-1)(x+4)}$	**+** above x-axis	**–** below x-axis	**+** above x-axis	**–** below x-axis	**+** Above x-axis

Draw sections of the graph through the zeroes indicating the correct transition based on the table of signs.

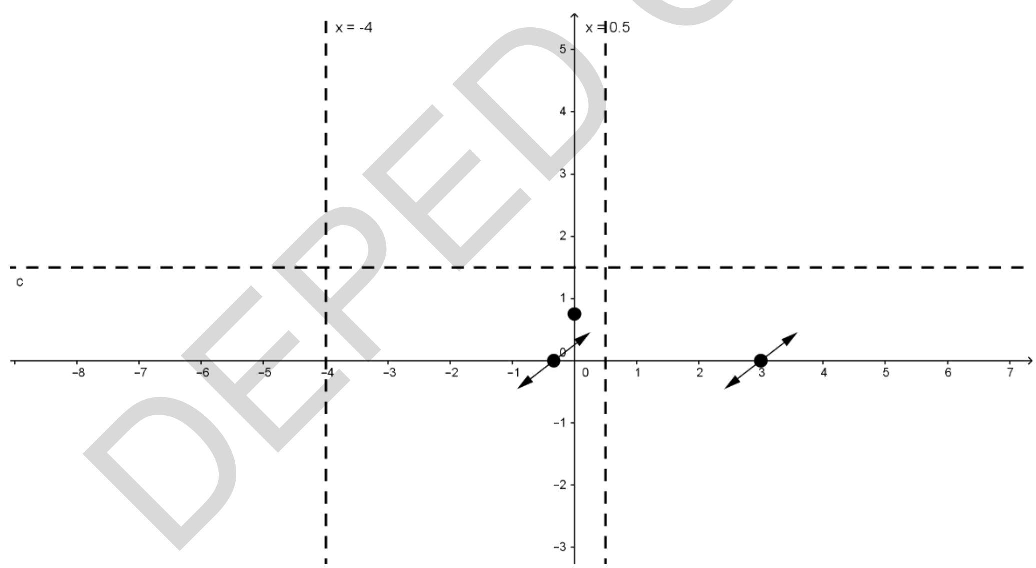


Figure 2.9: Sketch the transitions across the zeroes based on the table of signs

Draw sections of the graph near the asymptotes based on the transition indicated on the table of signs.

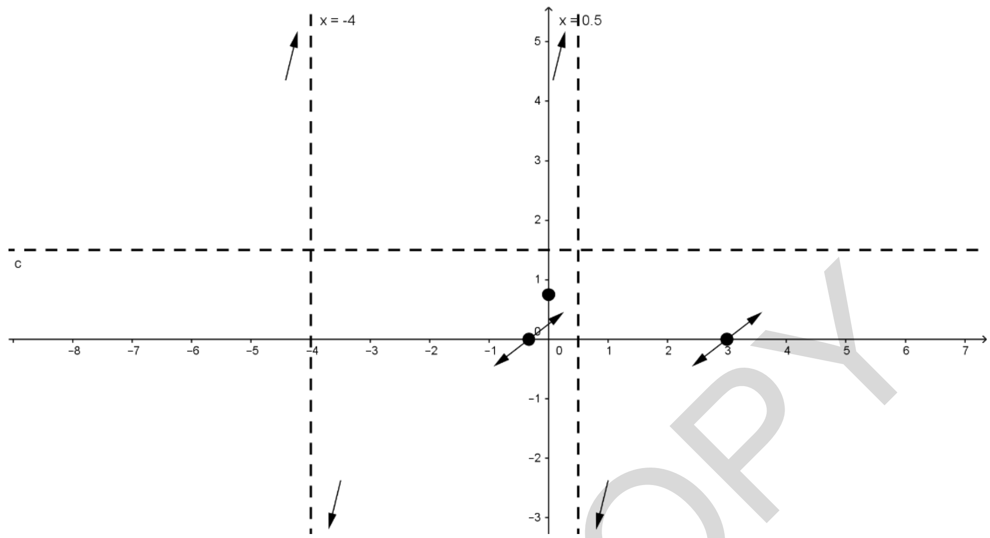


Figure 2.10: Sketch the graph near the asymptotes based on the table of signs.

Complete the sketch by connecting the arrowheads, making sure that the sketch passes through the y-intercept as well. The sketch should follow the horizontal asymptote as the x-values goes to the extreme left and right of the Cartesian plane.

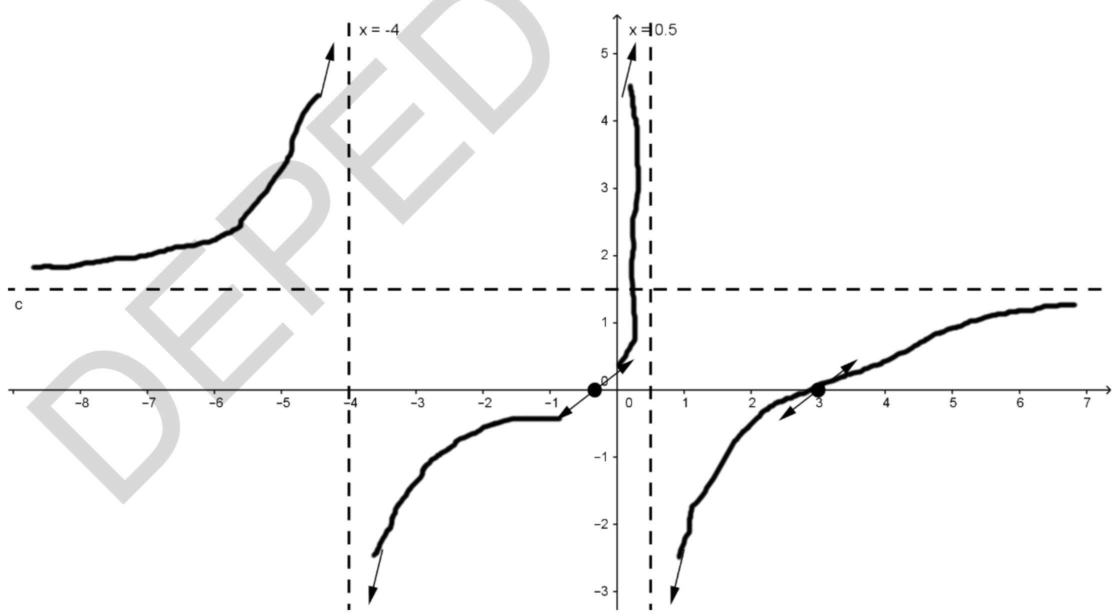


Figure 2.11: Rough sketch of the graph following the information above.

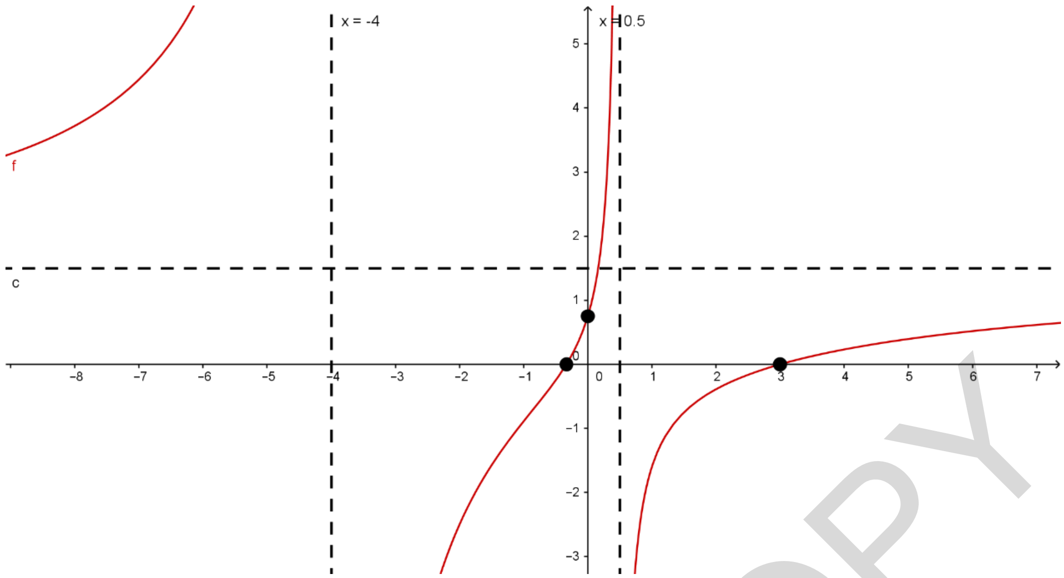


Figure 2.12: Actual sketch of the graph using a software grapher.

The domain of the function is all values of x not including those where the function is undefined. Therefore the domain of f(x) is $\left\{x \ \in \mathbb{R} \middle| \neq \frac{1}{2} \ and \ x \neq -4\right\}$.

From the graph of the function, we observe that the function increases and decreases without bound. The graph also crosses the horizontal asymptote. Therefore the range of the function is the set $\mathbb{R}$ of all real numbers.

Solved Examples

1. Let $f(x) = \frac{x+5}{x-1}$. (a) Find its domain, (b) intercepts, (c) asymptotes. Next, (d) sketch its graph and (e) determine its range.

Solution.

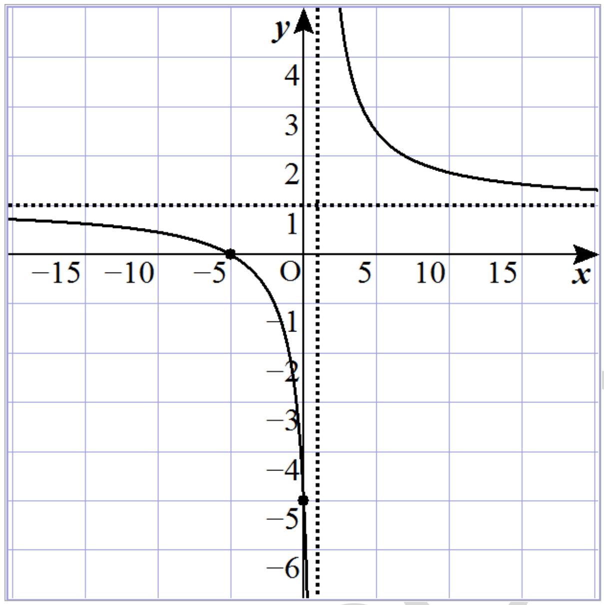
(a) The domain of f(x) is $\{x \ \in \mathbb{R} | x \ \neq 1\}$.
(b) The x-intercept is –5 and its y-intercept is –5.

(c) The vertical asymptote is x = 1. The degree of the numerator is equal to the degree of the denominator. The horizontal asymptote is y = 1/1 = 1.

(d) The table of signs is shown below.

Interval	$x < -5$	$-5 < x < 1$	$x > 1$
Test point	$x = -10$	$x = 0$	$x = 3$
x + 5	–	+	+
x – 1	–	–	+
$\frac{x+5}{x-1}$	**+** above x-axis	**–** below x-axis	**+** above x-axis

The graph of the function is given by:



(d) Based on the graph, the range of the function is $\{y \in \mathbb{R} | y \neq 1\}$.

2. Let $f(x) = \frac{x+3}{x^2-2x-3}$. Find its domain, (b) intercepts, (c) asymptotes. Next, (d) sketch its graph.

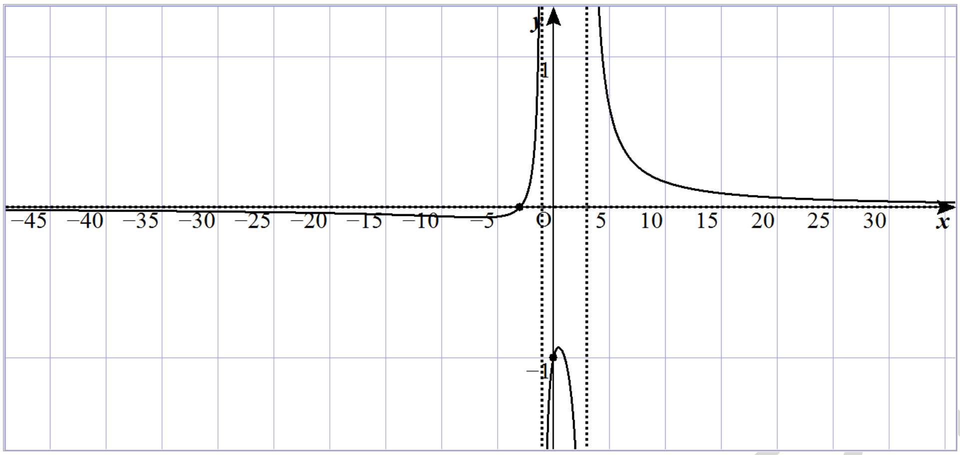
Solution. $f(x) = \frac{x+3}{x^2-2x-3} = \frac{x+3}{(x-3)(x+1)}$.

(a) The domain of f(x) is $\{x \in \mathbb{R} | x \neq -1\}$.
(b) The x-intercept is –3 and its y-intercept is –1.
(c) The vertical asymptotes are x = 3 and x = –1. The degree of the numerator is less than the degree of the denominator. The horizontal asymptote is y = 0.

(d) The table of signs is shown below.

Interval	$x < -3$	$-3 < x < -1$	$-1 < x < 3$	$x > 3$
Test point	$x = -10$	$x = -2$	$x = 0$	$x = 5$
x + 3	–	+	+	+
x – 3	–	–	–	+
x + 1	–	–	+	+
$\frac{x+5}{x-1}$	**–** below x-axis	**+** above x-axis	**–** below x-axis	**+** above x-axis

The graph of the function is given by:



Observe that the graph approaches the horizontal asymptote y = 0 as x increases or decreases without bound. Calculus is needed to determine the range of this function.

3.Past records from a factory suggest that new employees can assemble N(t) components per day after t days of being on the job, where $N(t) = \frac{75t}{t+5}, t \geq 0$. Sketch the graph of N. Identify the horizontal asymptote of N, and discuss its meaning in practical terms.

Solution.

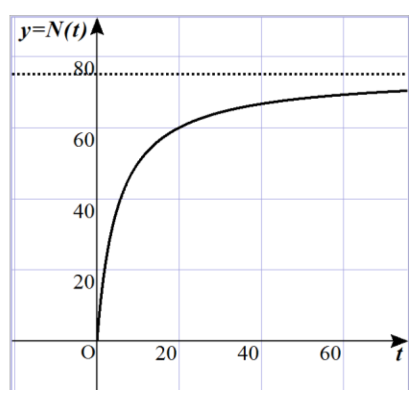
(a) The domain of N(t), as stated in the problem, is $\{t \in \mathbb{R} | t \geq 0\}$. (Negative values of t are not allowed because t refers to a number of days).

(b) The t-intercept is 0 and the y-intercept is 0.

(c) There is no vertical asymptote in the stated domain. The degree of the numerator and denominator are equal. The horizontal asymptote is y = 75.

(d) The table of signs is shown below.

Interval	$t > 0$
Test point	$t = 1$
75t	+
t + 5	+
$\frac{75t}{t+5}$	**+** above x-axis



As a person gains experience on the job, he or she works faster, but the maximum number of items that can be assembled cannot exceed 75.

Lesson 8 Supplementary Exercises

1. Find all asymptotes of $f(x) = \frac{x-5}{x^2-8x+12}$.

2. Explain why the function $f(x) = \frac{5x-15}{x-3}$ is not asymptotic to the line $x = 3$. Sketch the graph of this function.

3. Sketch the graph of $f(x) = \frac{2x-1}{x^2-4}$ and give its domain, intercepts, asymptotes, and range.

4. Sketch the graph of $f(x) = \frac{2x^2}{x^2+2x-3}$ and give its domain, intercepts, and asymptotes.

5. After a drug is injected into a patient's bloodstream, the concentration c of the drug in the bloodstream t minutes after the injection is given by $c(t) = \frac{20t}{t^2+2}, t \geq 0$. Sketch the graph of c. Identify the horizontal asymptote of c, and discuss its meaning in practical terms.

6. *A challenging riddle.* I am a rational function having a vertical asymptote at the lines x = 3 and x = –3, and a horizontal asymptote at y = 1. If my only x-intercept is 5, and my y-intercept is –5/9, what function am I?

Lessons 1- 8 Topic Test 1

1. True or False **[6]**

   (a) A function is a set of ordered pairs (x, y) such that no two ordered pairs have the same y-value but different x-values

   (b) The leading coefficient of $p(x) = 3x^2 - 4x^3 - x + 8$ is 3.

   (c) In a rational function, If the degree of the numerator is the smaller than the degree of the denominator then there is an asymptote at $x = 0$.

2. Give the domain of $y = \sqrt{4 - x^2}$ using set builder notation. **[4]**

3. Given $f(x) = 2x^2 - x - 6$, what is $f(2 - x)$? **[5]**

4. Given $f(x) = x - 4$ and $g(x) = 3x^2 - 8x - 16$, find: **[15]**

   (a) $(g - f)(x) - (f + g)(x)$

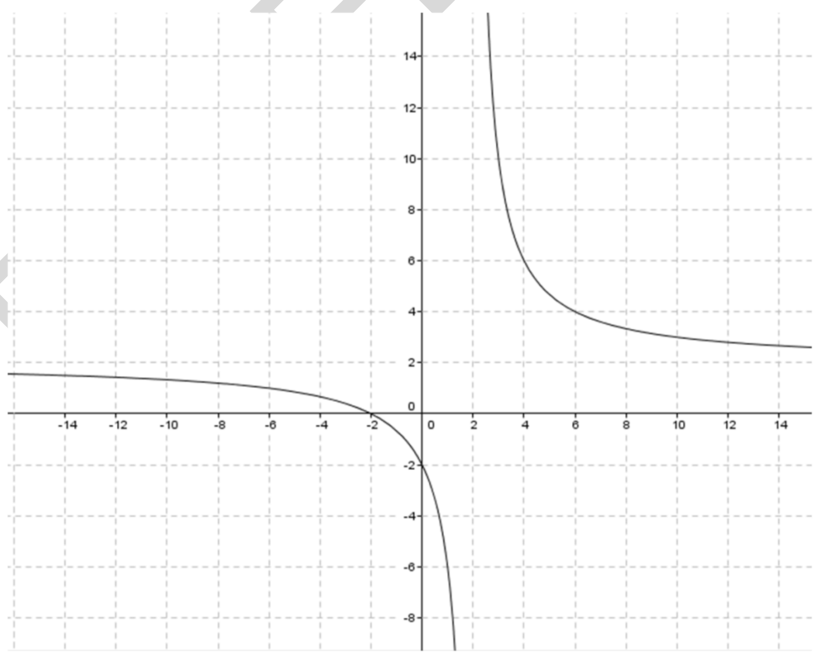
   (b) $g(x)/f(x)$

   (c) $(gof)(x)$

5. Is the solution set of $\frac{1}{x} \geq 1$ in set builder notation $\{x \in \mathbb{R} |\ x \leq 1\}$? Explain. **[5]**

6. Identify the zeroes of the function $f(x) = \frac{x^2+10x-24}{x^2-10x+24}$. For what values will the function be undefined? **[10]**

7. Identify the asymptotes of the graph below. **[5]**



Lessons 1 – 8 Topic Test 2

1. A part-time job gives you an hourly wage of P50.00. If you work for more than 40 hours per week, you get an overtime pay that is 1.5 times your normal hourly wage. Write a piecewise function $P(h)$ that gives your weekly pay in terms of the number of hours h you worked that week. **[10]**

2. Given the piecewise function $f(x) = \begin{cases} 2-x, & x < -3 \\ \sqrt{9-x^2}, & -3 \leq x < 3 \\ x^2 - 7, & x \geq 3 \end{cases}$, evaluate the function at the following values of x: **[5]**

   (a) $x = -2$
   (b) $x = 3$
   (c) $x = 0$
   (d) $x = -5$

3. Let $f(x) = \sqrt{x+5}$, $g(x) = x^3 - 4$, and $h(x) = 2x - 7$, find $(g \text{ o} (f + h))(4)$. **[10]**

4. Solve for x: $\frac{x}{x+4} + \frac{1}{x-3} = \frac{3}{x^2+x-12}$ **[10]**

5. Give the solution set of $\frac{3x-5}{x-5} \geq 4$ in set builder notation. **[10]**

6. Find the asymptotes of $f(x) = \frac{12+2x-4x^2}{2x^2-x-6}$. **[5]**

Lesson 9: One-to-One functions

Learning Outcome(s): At the end of the lesson, the learner is able to represent real-life situations using one-to-one functions.

Lesson Outline:

1. One-to-one functions
2. Examples of real-life situations represented by one-to-one functions.
3. Horizontal line test.

Definition: The function f is **one-to-one** if for any x_1, x_2 in the domain of f, then $f(x_1) \neq f(x_2)$. That is, the same y-value is never paired with two different x-values.

In Examples 1-5, determine whether the given relation is a function. If it is a function, determine whether it is one-to-one or not.

Example 1. The relation pairing an SSS member to his or her SSS number

Solution. Each SSS member is assigned to a unique SSS number. Thus, the relation is a function. Further, two different members cannot be assigned the same SSS number. Thus, the function is one-to-one.

Example 2. The relation pairing a real number to its square.

Solution. Each real number has a unique perfect square. Thus, the relation is a function. However, two different real numbers such as 2 and –2 may have the same square. Thus, the function is not one-to-one.

Example 3. The relation pairing an airport to its airport code

Airport codes are three letter codes used to uniquely identify airports around the world and prominently displayed on checked-in bags to denote the destination of these bags. Here are some examples of airport codes:

- MNL – Ninoy Aquino International Airport (All terminals)
- CEB – Mactan-Cebu International Airport
- DVO – Francisco Bangoy International Airport (Davao)
- JFK – John F. Kennedy International Airport (New York City
- CDG – Charles de Gaulle International Airport (Paris, France)

Airport codes can be looked up at https://www.world-airport-codes.com

Solution. Since each airport has a unique airport code, then the relation is a function. Also, since no two airports share the same airport code, then the function is one-to-one.

Example 4. The relation pairing a person to his or her citizenship.

Solution. The relation is not a function because a person can have dual citizenship (i.e., citizenship is not unique).

Example 5. The relation pairing a distance d (in kilometers) traveled along a given jeepney route to the jeepney fare for traveling that distance.

Solution. The relation is a function since each distance traveled along a given jeepney route has an official fare. In fact, as shown in Lesson 1, the jeepney fare may be represented by a piecewise function, as shown below:

$$F(d) = \begin{cases} 8.00 & \text{if } 0 < d \leq 4 \\ (8.00 + 1.50\lfloor d \rfloor) & \text{if } d > 4 \end{cases}$$

Note that $\lfloor d \rfloor$ is the floor or greatest integer function applied to d.

However, the function is not one-to-one because different distances (e.g., 2, 3 or 4 kilometers) are charged the same rate (P8.00). That is, because $F(3) = F(2) = F(3.5) = 8$, then F is not one-to-one.

A simple way to determine if a given graph is that of a one-to-one function is by using the Horizontal Line Test.

Horizontal Line Test. A function is one-to-one if each horizontal line does not intersect the graph at more than one point.

A graph showing the plot of $y = x^2 - 4$ fails the horizontal line test because some lines intersect the graph at more than one point.

The Vertical and Horizontal Line Tests. All functions satisfy the vertical line test. All one-to-one functions satisfy both the vertical and horizontal line tests.

Solved Examples

1. Which of the following are one-to-one functions?

   (a) Books to authors
   (b) SIM cards to cell phone numbers
   (c) True or False questions to answers

Solution.

Only b is a one-to-one function. Books can have multiple authors that wrote the book. A true or false question has only one answer so it is a function but a “True” answer can correspond to multiple questions.

2. Which of the following relations is a one-to-one function?

   (a) $\{(0,0), (1,1), (2,8), (3,27), (4,64)\}$
   (b) $\{(-2,4), (-1,1), (0,0), (1,1), (2,4)\}$
   (c) $\{(0,4), (1,5), (2,6), (3,7), \ldots\ (n, n+4), \ldots\}$

***Solution*.**

Both a and c are one-to-one functions. B is a function however it is not one-to-one since it has y-values that are paired up with two different x-values.

Lesson 9 Supplementary Exercises

1. For what values of k is the set of order pairs {(2, 4), (k, 6), (4, k)} a one-to-one function?

2. Consider each uppercase letter in the English alphabet as a graph. Is there any of these letters that will pass both the vertical and horizontal line tests?

3. The length of a rectangle, l, is four more than its width. Let $A(l)$ be the function mapping the length of the rectangle to its area. Is the function one-to-one?

Lesson 10: Inverse of One-to-One Functions

Learning Outcome(s): At the end of the lesson, the learner is able to determine the inverses of one-to-one functions.

Lesson Outline:

1. Inverse of a one-to-one function.
2. Finding the inverse of a one-to-one function.
3. Property of inverse functions

The importance of one-to-one functions is due to the fact that these are the only functions that have an inverse, as defined below.

Definition: Let f be a one-to-one function with domain A and range B. Then the **inverse** of f, denoted by f^{-1}, is a function with domain B and range A defined by $f^{-1}(y) = x$ if and only if $f(x) = y$ for any y in B.

A function has an inverse if and only if it is one-to-one. If a function f is not one-to-one, properly defining an inverse function f^{-1} will be problematic. For example, suppose that f(1) = 5 and f(3) = 5. If f^{-1} exists, then $f^{-1}(5)$ has to be both 1 and 3, and this prevents f^{-1} from being a valid function. This is the reason why the inverse is only defined for one-to-one functions.

To find the inverse of a one-to-one function:

(a) Write the function in the form y = f(x);
(b) Interchange the x and y variables;
(c) Solve for y in terms of x

Example 1. Find the inverse of $f(x) = 3x + 1$

Solution. The equation of the function is $y = 3x + 1$.

Interchange the x and y variables: $x = 3y + 1$

Solve for y in terms of x:

$$x = 3y + 1$$

$$x - 1 = 3y$$

$$\frac{x-1}{3} = y \ \Rightarrow y = \frac{x-1}{3}$$

Therefore the inverse of $f(x) = 3x + 1$ is $f^{-1}(x) = \frac{x-1}{3}$.

Property of an inverse of a one-to-one function

Given a one-to-one function $f(x)$ and its inverse $f^{-1}(x)$, then the following are true:

- The inverse of $f^{-1}(x)$ is $f(x)$.
- $f\left(f^{-1}(x)\right) = x$ for all x in the domain of f^{-1}.
- $f^{-1}\left(f(x)\right) = x$ for all x in the domain of f.

For the second and third properties above, it can be imagined that evaluating a function and its inverse in succession is like reversing the effect of the function. For example, the inverse of a function that multiplies 3 to a number and adds 1 is a function that subtracts 1 and then divides the result by 3.

Example 2. Find the inverse of $g(x) = x^3 - 2$.

Solution. The equation of the function is $y = x^3 - 2$.

Interchange the x and y variables: $x = y^3 - 2$.

Solve for y in terms of x:

$$x = y^3 - 2$$

$$x + 2 = y^3$$

$$y = \sqrt[3]{(x+2)}$$

The inverse of $g(x) = x^3 - 2$is $g^{-1}(x) = \sqrt[3]{(x+2)}$.

Example 3. Find the inverse of the rational function $f(x) = \frac{2x+1}{3x-4}$.

Solution. The equation of the function is: $y = \frac{2x+1}{3x-4}$

Interchange the x and y variables :$x = \frac{2y+1}{3y-4}$

Solve for y in terms of x:

$$x = \frac{2y+1}{3y-4}$$

$$x(3y-4) = 2y+1$$

$$3xy - 4x = 2y + 1$$

$$3xy - 2y = 4x + 1$$

(Place all terms with y on one side and those without y on the other side.)

$$y(3x-2) = 4x+1$$

$$y = \frac{4x+1}{3x-2}$$

Therefore the inverse of $f(x) = \frac{2x+1}{3x-4}$ is $f^{-1}(x) = \frac{4x+1}{3x-2}$.

Example 4. Find the inverse of $f(x) = x^2 + 4x - 2$, if it exists.

Solution. The students should recognize that this is a quadratic function with a graph in the shape of a parabola that opens upwards. It is not a one-to-one function as it fails the horizontal line test.

(*Optional*) We can still apply the procedure for finding the inverse of a one-to-one function to see what happens when it is applied to a function that is not one-to-one.

The equation of the function is: $y = x^2 + 4x - 2$

Interchange the x and y variables: $x = y^2 + 4y - 2$

Solve for y in terms of x

$$x = y^2 + 4y - 2$$

$$x + 2 = y^2 + 4y$$

$$x + 2 + 4 = y^2 + 4y + 4$$

(Complete the square)

$$x + 6 = (y+2)^2$$

$$\pm\sqrt{(x+6)} = y + 2$$

$$\pm\sqrt{(x+6)} - 2 = y \Rightarrow y = \pm\sqrt{(x+6)} - 2$$

The equation $y = \pm\sqrt{(x+6)} - 2$ does not represent a function because there are some x-values that correspond to two different y-values (e.g., if x = 3, y can be 1 or -5.). Therefore the function $f(x) = x^2 + 4x - 2$ has no inverse function.

Example 5. Find the inverse of $f(x) = |3x|$, if it exists.

Solution. Recall that the graph of $f(x) = |3x|$ is shaped like a "V" whose vertex is located at the origin. This function fails the horizontal line test and therefore has no inverse.

Alternate Solution. We can also show that f^{-1} does not exist by showing that f is not one-to-one. Note that f(1) = f(–1) = 3. Since the x-values 1 and –1 are paired to the same y-value, then f is not one-to-one and it cannot have an inverse.

(*Optional*) If we apply the procedure for finding the inverse of a one-to-one function:

The equation of the function is: $y = |3x|$

Interchange x and y: $x = |3y|$

Solve for y in terms of x:

$$x = |3y|$$

$$x = \sqrt{(3y)^2}$$

(Use $|x| = \sqrt{x^2}$)

$$x^2 = 3y$$

$$\frac{x^2}{3} = y$$

$$\pm\sqrt{\frac{x^2}{3}} = y \Rightarrow y = \pm\sqrt{\frac{x^2}{3}}$$

In this function, x=2 will correspond to f(x)=1 and f(x)= –1. Therefore f(x) has no inverse function.

Example 6. To convert from degrees Fahrenheit to Kelvin, the function is $k(t) = \frac{5}{9}(t - 32) + 273.15$, where *t* is the temperature in Fahrenheit (Kelvin is the SI unit of temperature). Find the inverse function converting the temperature in Kelvin to degrees Fahrenheit.

Solution. The equation of the function is: $k = \frac{5}{9}(t - 32) + 273.15$,

Since k and t refer to the temperatures in Kelvin and Fahrenheit respectively, we do not interchange the variables.

Solve for t in terms of k:

$$k = \frac{5}{9}(t - 32) + 273.15$$

$$k - 273.15 = \frac{5}{9}(t - 32)$$

$$\frac{9}{5}(k - 273.15) = t - 32$$

$$\frac{9}{5}(k - 273.15) + 32 = t \Rightarrow t = \frac{9}{5}(k - 273.15) + 32$$

Therefore the inverse function is $t(k) = \frac{9}{5}(k - 273.15) + 32$ where k is the temperature in Kelvin.

Solved Examples

1. Find the inverse of $f(x) = 2x + 7$

Solution.

$$y = 2x + 7$$

$$x = 2y + 7$$

$$x - 7 = 2y \Rightarrow \frac{x - 7}{2} = y \Rightarrow y = \frac{x - 7}{2}$$

Therefore, $f^{-1}(x) = \frac{x-7}{2}$.

2. Find the inverse of $f(x) = \frac{x-2}{3x+5}$

Solution.

$$y = \frac{x - 2}{3x + 5}$$

$$x = \frac{y - 2}{3y + 5}$$

$$3yx + 5x = y - 2$$

$$3yx - y = -5x - 2$$

$$y(3x - 1) = -5x - 2 \Rightarrow y = \frac{-5x - 2}{3x - 1}$$

Therefore, $f^{-1}(x) = \frac{-5x-2}{3x-1}$.

Lesson 10 Supplementary Exercises

1. Which among the following functions have an inverse?
   (a) $f(x) = 2x^3 - 5$
   (b) $g(x) = 3x - 8$
   (c) $h(x) = \frac{1}{x^2}$
   (d) $k(x) = |x|$
   (e) $l(x) = x^2 - 6x$
2. Find the inverse of $f(x) = -x^3 + 2$.
3. Find $f(x)$ if $f^{-1}(x) = \frac{1}{x-2}$.

Lesson 11: Graphs of Inverse Functions

Learning Outcome(s): At the end of the lesson, the learner is able to represent an inverse function through its table of values and graph, find the domain and range of an inverse function, graph inverse functions, solve problems involving inverse functions.

Lesson Outline:

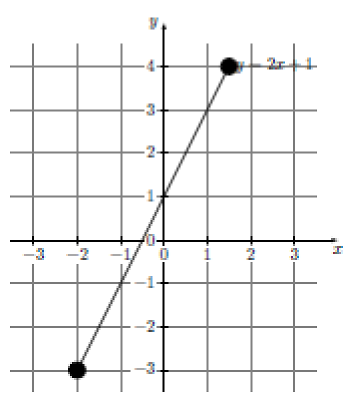
1. Graphs of inverse functions as a reflection about the line y = x
2. Domain and range of a one-to-one function and its inverse

Graphing Inverse Functions

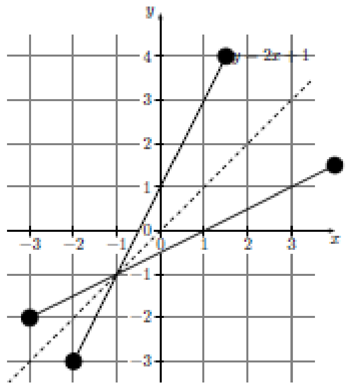
First we need to ascertain that the given graph corresponds to a one-to-one function by applying the horizontal line test. If it passes the test, the corresponding function is one-to-one.

Given the graph of a one-to-one function, the **graph of its inverse** can be obtained by reflecting the graph about the line $y = x$.

Example 1. Graph $y = f^{-1}(x)$ if the graph of $y = f(x) = 2x + 1$ restricted in the domain $\{x | -2 \leq x \leq 1.5\}$ is given below. What is the range of the function? What is the domain and range of its inverse?



Solution. Take the reflection of the restricted graph of $y = 2x + 1$ across the line $y = x$.



The range of the original function can be determined by the inspection of the graph. The range is $\{f(x) \in \mathbb{R} \mid -3 \leq f(x) \leq 4\}$.

Verify using techniques in an earlier lesson that the inverse function is given by

$$f^{-1}(x) = \frac{x-1}{2}.$$

The domain and range of the inverse function can be determined by inspection of the graph: Domain of $f^{-1}(x) = \{x \in \mathbb{R} \mid -3 \leq x \leq 4\}$

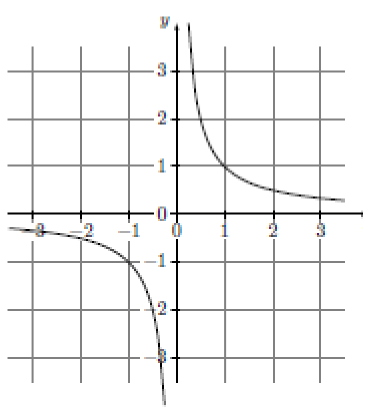
Range of $f^{-1}(x) = \{y \in \mathbb{R} \mid -2 \leq y \leq 1.5\}$

In summary,

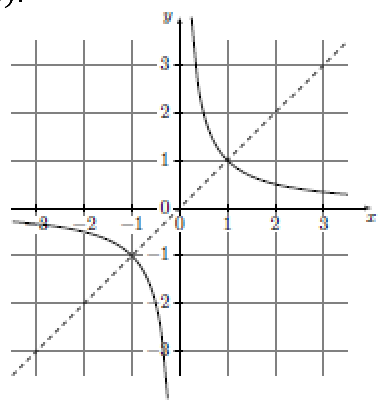
	$f(x)$	$f^{-1}(x)$
Domain	$\{x \in \mathbb{R} \mid -2 \leq x \leq 1.5\}$	$\{x \in \mathbb{R} \mid -3 \leq x \leq 4\}$
Range	$\{y \in \mathbb{R} \mid -3 \leq y \leq 4\}$	$\{y \in \mathbb{R} \mid -2 \leq y \leq 1.5\}$

Observe that the domain of the inverse is the range of the original function, and that the range f of the inverse is the domain of the original function. Is this true for all one-to-one functions and their inverses?

Example 2. Find the inverse of $f(x) = \frac{1}{x}$ using its given graph.

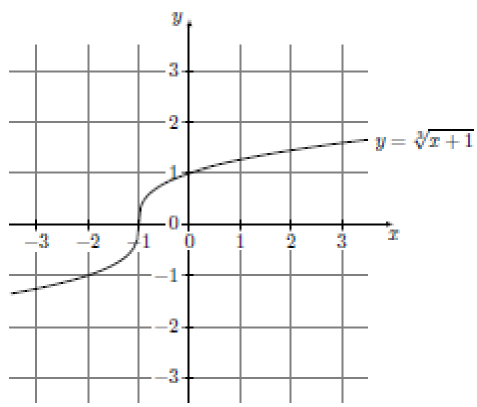


Solution. Applying the horizontal line test, we verify that the function is one-to-one. Since the graph of $f(x) = \frac{1}{x}$ is symmetric with respect to the line $y = x$ (indicated by a dashed line), its reflection across the line $y = x$ is itself. Therefore the inverse of $f(x)$ is itself or $f^{-1}(x) = f(x)$.

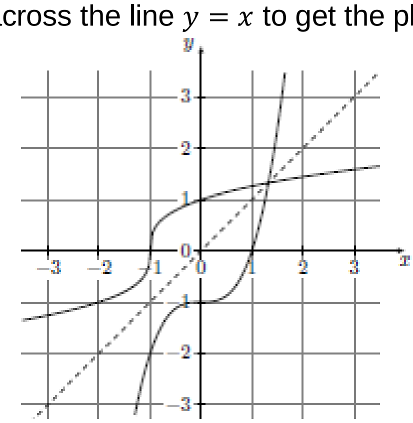


Verify that $f^{-1}(x) = f(x) = \frac{1}{x}$ using the techniques used in the previous lesson.

Example 3. Find the inverse of $f(x) = \sqrt[3]{x+1}$ using the given graph.

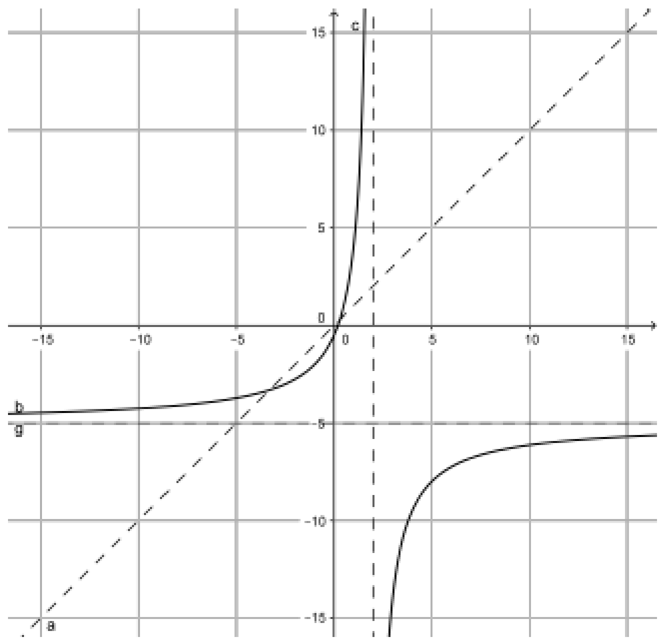


Solution. Applying the horizontal line test, we confirm that the function is one-to-one. Reflect the graph of $f(x)$ across the line $y = x$ to get the plot of the inverse function.



The result of the reflection of the graph of $f(x) = \sqrt[3]{x+1}$ is the graph of $y = x^3 - 1$. Therefore, $f^{-1}(x) = x^3 - 1$.

Example 4. Consider the rational function $f(x) = \frac{5x-1}{-x+2}$ whose graph is shown below:



(a) Find its domain and range.
(b) Find the equation of its asymptotes.
(c) Find the graph of its inverse.
(d) Find the domain and range of its inverse.

Solution.

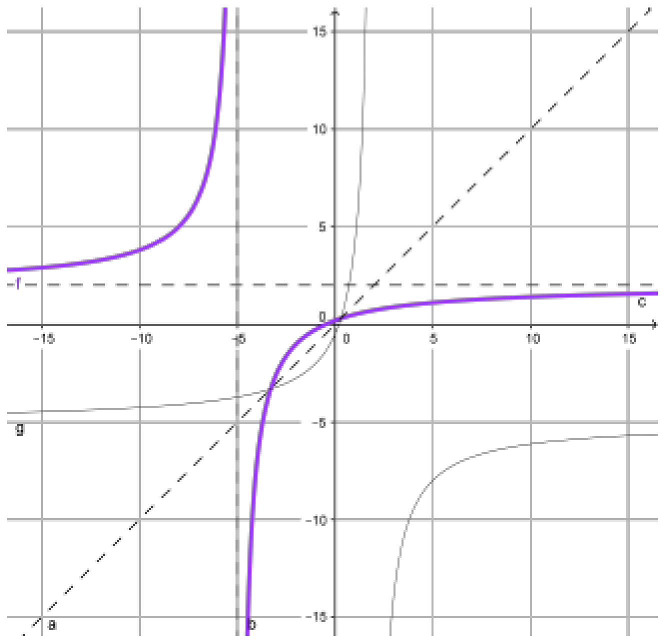
(a) From our lessons on rational functions, we get the following results:

$$\text{Domain of } f(x) = \{x \in \mathbb{R} \,|\, x \neq 2\}$$
$$\text{Range of } f(x) = \{y \in \mathbb{R} \,|\, y \neq -5\}$$

(b) Using techniques from the lesson on rational functions, the equations of the asymptotes are

Vertical asymptote: $x = 2$
Horizontal asymptote: $y = -5$

(c) The inverse can be graphed by taking the reflection of the graph across $y = x$



Observe that the new asymptotes are the old asymptotes with the x and y values interchanged. In fact, the asymptotes could also be obtained by reflecting the original asymptotes about the line $y = x$.

Vertical asymptote: $x = -5$

Horizontal asymptote: $y = 2$

(d) The domain and range of the functions and its inverse are as follows:

	$f(x)$	$f^{-1}(x)$
Domain	$\{x \in \mathbb{R} \mid x \neq 2\}$	$\{x \in \mathbb{R} \mid x \neq -5\}$
Range	$\{y \in \mathbb{R} \mid y \neq -5\}$	$\{y \in \mathbb{R} \mid y \neq 2\}$

We can make the observation that the domain of the inverse is the range of the original function and the range of the inverse is the domain of the original function.

Example 5. In the examples above, what will happen if we plot the inverse functions of the inverse functions?

Solution. If we plot the inverse of a function, we reflect the original function about the line $y = x$. If we plot the inverse of the inverse, we just reflect the graph back about the line $y = x$ and end up with the original function.

This result implies that the original function is the inverse of its inverse, or $(f^{-1})^{-1}(x) = f(x)$.

Solving problems involving inverse functions

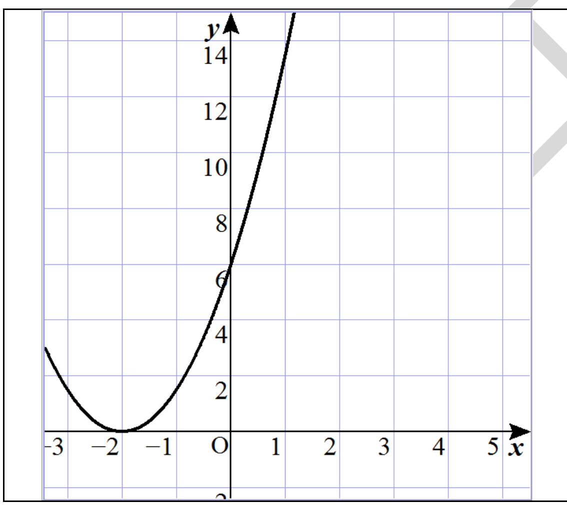
We can apply the concepts of inverse functions in solving word problems involving reversible processes.

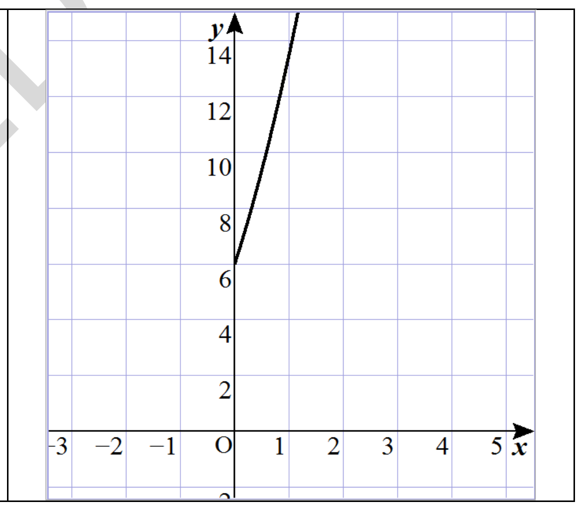
Example 6. You asked a friend to think of a nonnegative number, add two to the number, square the number, multiply the result by 3 and divide the result by 2. If the result is 54, what is the original number? Construct an inverse function that will provide the original number if the result is given.

Solution. We first construct the function that will compute the final number based on the original number. Following the instructions, we come up with this function:

$$f(x) = (x+2)^2 \cdot 3 \div 2 = \frac{3(x+2)^2}{2}$$

The graph is shown below, on the left. This is *not* a one-to-one function because the graph does not satisfy the horizontal line test. However, the instruction indicated that the original number must be nonnegative. The domain of the function must thus be restricted to $x \geq 0$, and its graph is shown on the right, below.





The function with restricted domain $x \geq 0$ is then a one-to-one function, and we can find its inverse.

Interchange the x and y variables: $x = \frac{3(y+2)^2}{2}, y \geq 0$

Solve for y in terms of x:

$$x = \frac{3(y+2)^2}{2}$$

$$\frac{2x}{3} = (y+2)^2$$

$$\sqrt{\frac{2x}{3}} = y + 2$$ (Since $y \geq 2$ we do not need to consider $-\sqrt{\frac{2x}{3}}$)

$$\sqrt{\frac{2x}{3}} - 2 = y \Rightarrow y = \sqrt{\frac{2x}{3}} - 2 \quad \Rightarrow \quad f^{-1}(x) = \sqrt{\frac{2x}{3}} - 2$$

Finally we evaluate the inverse function at $x = 54$ to determine the original number:

$$f^{-1}(54) = \sqrt{\frac{2(54)}{3}} - 2 = \sqrt{\frac{108}{3}} - 2 = \sqrt{36} - 2 = 6 - 2 = 4$$

The original number is 4.

Example 7. Engineers have determined that the maximum force t in tons that a particular bridge can carry is related to the distance d in meters between it supports by the following function:

$$t(d) = (12.5/d)^3$$

How far should the supports be if the bridge is to support 6.5 tons? Construct an inverse function to determine the result.

Solution. The equation of the function is $t = (12.5/d)^3$.

To lessen confusion in this case, let us not interchange d and t as they denote specific values. Solve instead for d in terms of t:

$$t = (12.5/d)^3$$
$$\sqrt[3]{t} = 12.5/d$$
$$d = 12.5/\sqrt[3]{t}$$

The inverse function is $d(t) = 12.5/\sqrt[3]{t}$.

Evaluate the function at t=6.5: $d(6.5) = 12.5/\sqrt[3]{6.5}$ = 6.70.

The supports should be placed at most 6.70 meters apart.

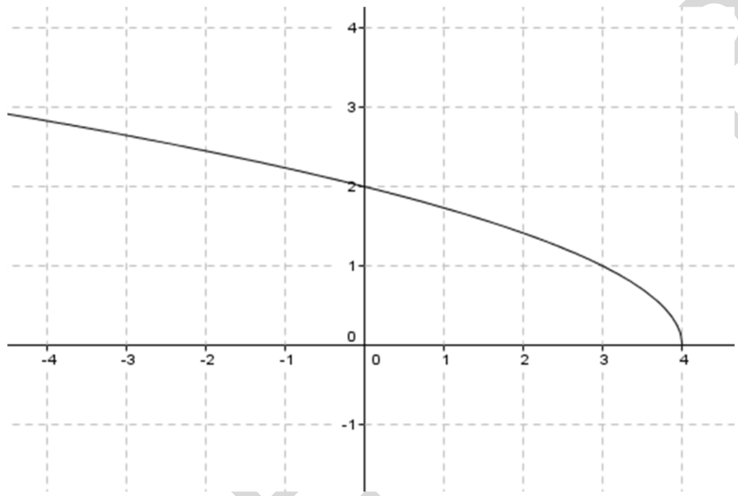
Solved Examples

1. If $f(x) = \sqrt{4-x}$ is restricted on the domain $\{x \in \mathbb{R} | -5 \leq x < 3\}$, what is the domain of its inverse?

***Solution*.**

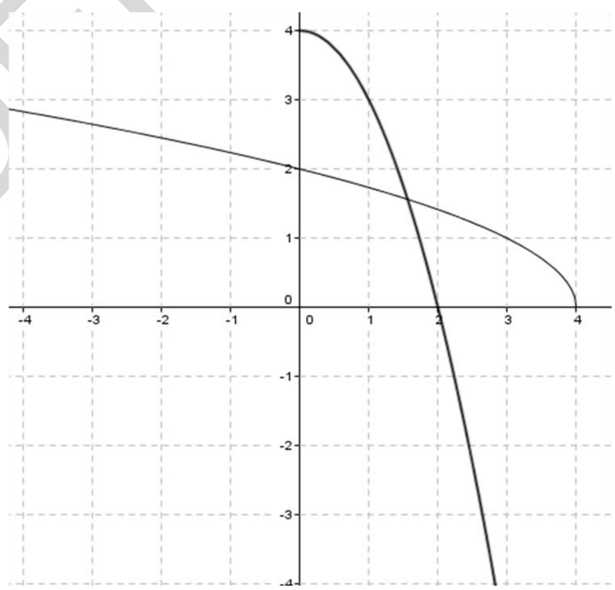
The domain of the inverse of $f(x)$ is just the range of $f(x)$. The range of $f(x)$ is $\{f(x) \in \mathbb{R} | 1 < f(x) \leq 3\}$. Therefore the domain of $f^{-1}(x)$ is $\{x \in \mathbb{R} | 1 < x \leq 3\}$

2. Given the graph of $f(x) = \sqrt{4-x}$ below, sketch the graph of its inverse.



***Solution*.**

The graph of $f^{-1}(x)$ is just the graph's reflection along y = x. So we get:



3. Using algebraic methods, construct the inverse of $f(x) = \sqrt{4-x}$. Is the function you get the same as the sketch of the inverse in the previous number?

***Solution*.**

To get $f^{-1}(x)$, we first interchange x and y in $y = \sqrt{4-x}$. So we get $x = \sqrt{4-y}$.

We then isolate y

$$x^2 = 4 - y$$

$$y = 4 - x^2$$

So we get $f^{-1}(x) = 4 - x^2$. However, the graph of that will result in a parabola opening downwards while the sketch we have in number 2 was just half that parabola. This occurs because the function must be one-to-one to have an inverse.

Lesson 11 Supplementary Exercises

1. Find the domain and range of the inverse of $f(x) = x^2 - 6x + 5$ with domain restriction $\{x \in \mathbb{R} |\ 0 < x < 3\}$.

2. Give the vertical and horizontal asymptotes of $f(x) = \frac{x+3}{x+1}$. Give the vertical and horizontal asymptotes of its inverse.

3. At what point will the graph of $f(x) = 2x - 5$ and its inverse intersect?

4. The formula for converting Celsius to Fahrenheit is given as $F = \frac{9}{5}C + 32$ where C is the temperature in Celsius and F is the temperature in Fahrenheit. Find the formula for converting Fahrenheit to Celsius. If the temperature in a thermometer reads 101.3°F, what is that in °C?

5. A particular breed of tilapia has its weight w (in kg) related to its length L (in cm) modeled by the function w = $(3.24 \times 10^{-3})L^2$. Explain why the function is one-to-one, even if it is a quadratic function. Find the inverse of this function and approximate the length of a single fish if its weight is 400 grams.

Lessons 9 – 11 Topic Test 1

1. True or False **[6]**

   (a) A linear function is a one-to-one function.
   (b) The inverse of $y = \frac{1}{x}$ is $y = x$.
   (c) The graph of the inverse of a function can be obtained by reflecting the graph of the function along $y = x$.

2. Identify if the following are one-to-one functions or not. **[6]**

   (a) People to their birthdays
   (b) People to their Social Security System number
   (c) People to their place of residence

3. Which of the following functions have an inverse function? If so, find its inverse. **[18]**

   (a) $f(x) = x^3 - 1$
   (b) $g(x) = \sqrt{x - 3}$
   (c) $p(x) = x^2 - 3$
   (d) $q(x) = |\mathrm{x} - 3|$
   (e) $r(x) = \frac{3}{x-1}$

4. Sketch the graph of the inverse of the function $f(x) = \sqrt{x + 3}$. **[10]**

Lessons 9 – 11 Topic Test 2

1. Find the inverse of the following functions: **[15]**

   (a) $f(x) = 7x + 4$
   (b) $g(x) = (x - 3)^3$

2. Find the domain and range of the inverse of $f(x) = \sqrt{x + 3}$ **[10]**

3. Find the asymptotes of the inverse of $q(x) = \frac{12x+5}{6-4x}$ **[10]**

4. At what point/s do the following functions and their inverses intersect? **[15]**

   (a) $f(x) = 3x - 8$
   (b) $g(x) = \sqrt{x + 2}$
   (c) $h(x) = x^3$

Lesson 12: Representing Real-Life Situations Using Exponential Functions

Learning Outcome(s): At the end of the lesson, the learner is able to represent real-life situations using exponential functions.

Lesson Outline:

1. Exponential functions
2. Population, half-life, compound interest
3. Natural exponential function

Definition: An **exponential function with base b** is a function of the form $f(x) = b^x$ or $y = b^x (b > 0, \quad b \neq 1)$.

Example 1. Complete a table of values for x = –3, –2, –1, 0, 1, 2, and 3 for the exponential functions $y = (1/3)^x$, $y = 10^x$, and $y = (0.8)^x$.

Solution.

x	–3	–2	–1	0	1	2	3
$y = (1/3)^x$	27	9	3	1	1/3	1/9	1/27
$y = 10^x$	1/1000	1/100	1/10	1	10	100	1000
$y = (0.8)^x$	1.953125	1.5625	1.25	1	0.8	0.64	0.512

Example 2. If $f(x) = 3^x$, evaluate f(2), f(–2), f(1/2), f(0.4), and f(π).

Solution.

$f(2) = 3^2 = 9$

$f(-2) = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

$f(1/2) = 3^{1/2} = \sqrt{3}$

$f(0.4) = 3^{0.4} = 3^{2/5} = \sqrt[5]{3^2} = \sqrt[5]{9}$

Since π = 3.14159… is irrational, the rules for rational exponents are not applicable. We define 3^π using rational numbers: $f(\pi) = 3^\pi$ can be approximated by $3^{3.14}$. A better approximation is $3^{3.14159}$. Intuitively, one can obtain any level of accuracy for 3^π by considering sufficiently more decimal places of π. Mathematically, it can be proved that these approximations approach a unique value, which we define to be 3^π.

Definition. Let b be a positive number not equal to 1. A **transformation** of an exponential function with base b is a function of the form

$$g(x) = a \cdot b^{x-c} + d,$$

where a, c, and d are real numbers.

Some of the most common applications in real-life of exponential functions and their transformations are population growth, exponential decay, and compound interest.

Example 3. At t = 0, there were initially 20 bacteria. Suppose that the bacteria doubles every 100 hours. Give an exponential model for the bacteria as a function of t.

Solution.

Initially,	at	$t = 0$	Number of bacteria = 20
	at	$t = 100$	Number of bacteria = $20(2)$
	at	$t = 200$	Number of bacteria = $20(2)^2$
	at	$t = 300$	Number of bacteria = $20(2)^3$
	at	$t = 400$	Number of bacteria = $20(2)^4$

An exponential model for this situation is $y = 20(2)^{t/100}$.

Exponential Models and Population Growth
Suppose a quantity y doubles every T units of time. If y_0 is the initial amount, then the quantity y after t units of time is given by $y = y_0(2)^{t/T}$.

The **half-life** of a radioactive substance is the time it takes for half of the substance to decay.

Example 4. Suppose that the half-life of a certain radioactive substance is 10 days and there are 10g initially, determine the amount of substance remaining after 30 days, and give an exponential model for the amount of remaining substance.

Solution*.* We use the fact that the mass is halved every 10 days (from definition of half-life).

Let t = time in days. Thus, we have:

Initially,	at	t= 0	Amount of Substance = 10g
	at	t=10 days	Amount of Substance = 5g
	at	t=20 days	Amount of Substance = 2.5g
	at	t=30 days	Amount of Substance = 1.25g

An exponential model for this situation is $y = 10(1/2)^{t/10}$.

Exponential Functions and Half-life
If the half-life of a substance is T units, and y_0 is the amount of the substance corresponding to t = 0, then the amount y of substance remaining after t units of time is given by $y = y_0(1/2)^{t/T}$.

A starting amount of money (called the **principal**) can be invested at a certain interest rate that is earned at the end of a given period of time (such as one year). If the interest rate is **compounded**, the interest earned at the end of the period is added to the principal, and this new amount will earn interest in the next period. The same process is repeated for each succeeding period: interest previously earned will also earn interest in the next period.

Example 5. Mrs. De la Cruz invested P100,000.00 in a company that offers 6% interest compounded annually. Define an exponential model for this situation. How much will this investment be worth at the end of each year for the next five years?

Solution.

Initially,	at $t = 0$	Investment = **P100,000**
	at $t = 1$	Investment = P100,000(1.06) = **P106,000**
	at $t = 2$	Investment = P106,000(1.06) = **P112,360**
	at $t = 3$	Investment = P112,360(1.06) ≈ **P119,101.60**
	at $t = 4$	Investment = P119,101.60(1.06) ≈ **P126,247.70**
	at $t = 5$	Investment = P 26,247.70(1.06) ≈ **P133,822.56**

An exponential model for this situation is $y = 100{,}000(1.06)^t$. The investment is worth P133,822.56.

Compound Interest.
If a principal P is invested at an annual rate of r, compounded annually, then the amount after t years is given by $A = P(1 + r)^t$.

Example 6. Referring to Example 5, is it possible for Mrs. De la Cruz to double her money in 8 years? in 10 years?

Solution. Using the model $y = 100{,}000(1.06)^t$, substitute $t = 8$ and $t = 10$:

If $t = 8$, $y = \text{P}100{,}000(1.06)^8 \approx \text{P}159{,}384.81$

If $t = 10$, $y = \text{P}100{,}000(1.06)^{10} \approx \text{P}179{,}084.77$

Since her money still has not reached P200,000 after 10 years, then she has not doubled her money during this time.

The Natural Exponential Function

While an exponential function may have various bases, a frequently used based is the irrational number *e*, whose value is approximately 2.71828. The enrichment in Lesson 27 will show how the number *e* arises from the concept of compound interest. Because *e* is a commonly used based, the natural exponential function is defined having *e* as the base.

Definition

The **natural exponential function** is the function $f(x) = e^x$.

Example 7. A large slab of meat is taken from the refrigerator and placed in a pre-

heated oven. The temperature T of the slab t minutes after being placed in the oven is given by $T = 170 - 165e^{-0.006t}$ degrees Celsius. Construct a table of values for the following values of t: 0, 10, 20, 30, 40, 50, 60, and interpret your results. Round off values to the nearest integer.

Solution.

t	0	10	20	30	40	50	60
T	5	15	24	32	40	47	54

The slab of meat is increasing in temperature at roughly the same rate.

Solved Examples

1. Robert invested P30,000 after graduation. If the average interest rate is 5.8% compounded annually, **(a)** give an exponential model for the situation, and **(b)** will the money be doubled in 15 years?

***Solution*.**

(a) At t = 0, the amount is P30,000.
At t = 1, the amount is P30,000(1.058) = P31,740.
At t = 2, the amount is P30,000$(1.058)^2$ = P33,580.92
At t = 3, the amount is P30,000$(1.058)^3$ = P35,528.61.
An exponential model for this situation is $y = 30000\ (1.058)^t$.

(b) If t = 15, then y = 69,888.59. The money has more than doubled in 15 years.

2. At time t = 0, 500 bacteria are in a petri dish, and this amount triples every 15 days. **(a)** Give an exponential model for the situation. **(b)** How many bacteria are in the dish after 40 days?

***Solution*.**

(a) Let y be the number of bacteria.
At t = 0, y = 500.
At t = 15, y = 500(3) = 1,500.
At t = 30, $y = 500(3)^2$ = 4,500.
At t = 45, $y = 500(3)^3$ = 13,500.
At t = 60, $y = 500(3)^4$ = 40,500.
An exponential model for this situation is $y = 500(3)^{t/15}$.

(b) If t = 40, then $y = 500(3)^{40/15} \approx 9360$. There will be 9360 bacteria after 40 days.

3. The half-life of a substance is 400 years. **(a)** Give an exponential model for the situation. **(b)** How much will remain after 600 years if the initial amount was 200 grams?

Solution.

(a) At t = 0, the amount is 200 grams.
At t = 400, the amount is 200(1/2) = 100.
At t = 800, the amount is $200(1/2)^2 = 50$
At t = 1200, the amount is $200(1/2)^3 = 25$
Thus, an exponential model for this situation is $y = 200(1/2)^{t/400}$.

(b) If t = 600, then y = 70.71 grams.

4. The population of the Philippines can be approximated by the function P(x) = $20000000e^{0.0251x}$ ($0 \le x \le 40$) where *x* is the number of years since 1955 (e.g. *x* = 0 at 1955). Use this model to approximate the Philippine population during the years 1955, 1965, 1975, and 1985. Round of answers to the nearest thousand.

Solution.

t	0	10	20	30
T	20,000,000	25,706,000	33,040,000	42,467,211

Lesson 12 Supplementary Exercises

1. A barangay has 1,000 individuals and its population doubles every 60 years. Give an exponential model for the barangay. What is the barangay's population in 10 years?

2. A bank offers a 2% annual interest rate, compounded annually, for a certain fund. Give an exponential model for a sum of P10,000 invested under this scheme. How much money will there be in the account after 20 years?

4. The half-life of a radioactive substance is 1200 years. If the initial amount of the substance is 300 grams, give an exponential model for the amount remaining after *t* years. What amount of substance remains after 1000 years?

Lesson 13: Exponential Functions, Equations, and Inequalities

Learning Outcome(s): At the end of the lesson, the learner is able to distinguish among exponential functions, exponential equations and exponential inequality.

Lesson Outline:

1. Exponential functions, exponential equations and exponential inequalities

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Definition: An **exponential expression** is an expression of the form $a \cdot b^{x-c} + d$, where $(b > 0, \ b \neq 1)$.

The definitions of exponential equations, inequalities and functions are shown below.

	Exponential Equation	Exponential Inequality	Exponential Function
Definition	An equation involving exponential expressions	An inequality involving exponential expressions	Function of the form $f(x) = b^x$ $(b > 0, b \neq 1)$
Example	$7^{2x-x^2} = \frac{1}{343}$	$5^{2x} - 5^{x+1} \leq 0$	$f(x) = (1.8)^x$ or $y = (1.8)^x$

An exponential equation or inequality can be solved for all x values that satisfy the equation or inequality (Lesson 13).An exponential function expresses a relationship between two variables (such as x and y), and can be represented by a table of values or a graph (Lessons 14 and 15).

Solved Examples

Determine whether the given is an exponential function, an exponential equation, an exponential inequality, or none of these.

1. $f(x) = 5x^2$ (Answer: None of these)
2. $2 \geq (1/2)^x$ (Answer: Exponential inequality)
3. $7^{4x} = y$ (Answer: Exponential function)
4. $4(10^{x-2}) = 500$ (Answer: Exponential equation)
5. $7 < 14^{x+3}$ (Answer: Exponential inequality)
6. $y = 0.5^x$ (Answer: Exponential function)

Lesson 13 Supplementary Exercises

Determine whether the given is an exponential function, an exponential equation, an exponential inequality or none of these.

1. $49^x = 7^2$
2. $3 < 9^x$
3. $y = 81^x$
4. $3(15x) = 45$
5. $3 \geq 9^{x-1}$
6. $y = 1.25^x$

Lesson 14: Solving Exponential Equations and Inequalities

Learning Outcome(s): At the end of the lesson, the learner is able to solve exponential equations and inequalities, and solve problems involving exponential equations and inequalities

Lesson Outline:

1. Solve exponential equations
2. Solve exponential inequalities

One-to-one Property of Exponential Functions
If $x_1 \neq x_2$, then $b^{x_1} \neq b^{x_2}$. Conversely, if $b^{x_1} = b^{x_2}$ then $x_1 = x_2$.

Example 1. Solve the equation $4^{x-1} = 16$.

Solution. Write both sides with 4 as the base.

$4^{x-1} = 16$

$4^{x-1} = 4^2$

$x - 1 = 2$

$x = 2 + 1$

$x = 3$

Alternate Solution. Write both sides with 2 as the base.

$4^{x-1} = 16$

$(2^2)^{x-1} = 2^4$

$2^{2(x-1)} = 2^4$

$2(x - 1) = 4$

$2x - 2 = 4$

$2x = 6$

$x = 3$

Example 2. Solve the equation $125^{x-1} = 25^{x+3}$.

Solution. Both 125 and 25 can be written using 5 as the base.

$125^{x-1} = 25^{x+3}$

$(5^3)^{x-1} = (5^2)^{x+3}$

$5^{3(x-1)} = 5^{2(x+3)}$

$3(x - 1) = 2(x + 3)$

$3x - 3 = 2x + 6$

$x = 9$

Example 3. Solve the equation $9^{x^2} = 3^{x+3}$.

Solution. Both 9 and 3 can be written using 3 as the base.

$(3^2)^{x^2} = 3^{x+3}$

$3^{2x^2} = 3^{x+3}$

$2x^2 = x + 3$

$2x^2 - x - 3 = 0$

$(2x - 3)(x + 1) = 0$

$2x - 3 = 0$ or $x + 1 = 0$

$x = 3/2$ or $x = -1$

Property of Exponential Inequalities

If $b > 1$, then the exponential function $y = b^x$ is increasing for all x. This means that $b^x < b^y$ if and only if $x < y$.

If $0 < b < 1$, then the exponential function $y = b^x$ is decreasing for all x. This means that $b^x > b^y$ if and only if $x < y$.

Example 4. Solve the inequality $3^x < 9^{x-2}$.

Solution. Both 9 and 3 can be written using 3 as the base.

$3^x < (3^2)^{x-2}$

$3^x < 3^{2(x-2)}$

$3^x < 3^{2x-4}$

Since the base 3 > 1, then this inequality is equivalent to

$x < 2x - 4$ (the direction of the inequality is retained)

$4 < 2x - x$

$4 < x$

The solution set to the inequality is $\{x \in \mathbb{R} | \ x > 4\}$.

Example 5. Solve the inequality $\left(\frac{1}{10}\right)^{x+5} \geq \left(\frac{1}{100}\right)^{3x}$.

Solution. Since $\frac{1}{100} = \left(\frac{1}{10}\right)^2$, then we write both sides of the inequality with $\frac{1}{10}$ as the base.

$\left(\frac{1}{10}\right)^{x+5} \geq \left(\frac{1}{100}\right)^{3x}$

$\left(\frac{1}{10}\right)^{x+5} \geq \left(\frac{1}{10^2}\right)^{3x}$

$\left(\frac{1}{10}\right)^{x+5} \geq \left(\frac{1}{10}\right)^{6x}$

Since the base $\frac{1}{10} < 1$, then this inequality is equivalent to

$x + 5 \leq 6x$ (the direction of the inequality is reversed)

$5 \leq 6x - x$

$5 \leq 5x$

$1 \leq x$

The solution set is $\{x \in \mathbb{R} \mid x \geq 1\}$.

Example 6. The half-life of Zn-71 is 2.45 minutes.[2] At t = 0, there were y_0 grams of Zn-71, but only $\frac{1}{256}$ of this amount remains after some time. How much time has passed?

Solution. Using exponential models in Lesson 12, we can determine that after t minutes, the amount of Zn-71 in the substance is $y_0\left(\frac{1}{2}\right)^{t/2.45}$.

We solve the equation $y_0\left(\frac{1}{2}\right)^{t/2.45} = \frac{1}{256}y_0$.

$$\left(\frac{1}{2}\right)^{t/2.45} = \frac{1}{256}$$

$$\left(\frac{1}{2}\right)^{t/2.45} = \left(\frac{1}{2}\right)^8$$

$$\frac{t}{2.45} = 8 \Rightarrow \quad t = 19.6$$

Thus, 19.6 minutes have passed since t = 0.

[2] http://www.periodictable.com/Isotopes/030.71/index.p.full.html

Solved Examples

Solve for x in the following equations or inequalities.

1. $3^x = 81$

Solution.

$3^x = 3^4$

$x = 4$

2. $5^{7-x} = 125$

Solution.

$5^{7-x} = 5^3$

$7 - x = 3$

$-x = 3 - 7$

$-x = -4$

$x = 4$

3. $\left(\frac{4}{6}\right)^x \geq \frac{36}{16}$

Solution.

$\left(\frac{4}{6}\right)^x \geq \frac{6^2}{4^2}$

$\left(\frac{4}{6}\right)^x \geq \left(\frac{6}{4}\right)^2$

$\left(\frac{4}{6}\right)^x \geq \left(\left(\frac{4}{6}\right)^{-1}\right)^2$

$\left(\frac{4}{6}\right)^x \geq \left(\frac{4}{6}\right)^{-2}$

$\{x \in \mathbb{R} \mid x \leq -2\}$

4. $5^x > 25^{x+1}$

Solution.

$5^x > (5^2)^{x+1}$

$5^x > 5^{2x+2}$

$x > 2x + 2$

$x - 2x > 2$

$-x > 2$

$\{x \in \mathbb{R} \mid x < -2\}$

5. At time t = 0, 480 grams of an isotope with a half life of 30 hours is present. How much time will have elapsed when only 15 grams remain?

Solution. The amount of substance after t hours $480\left(\frac{1}{2}\right)^{t/30}$.

We solve the equation $480\left(\frac{1}{2}\right)^{t/30} = 15$

$$\left(\frac{1}{2}\right)^{t/30} = \frac{15}{480} = \frac{1}{32}$$

$$\left(\frac{1}{2}\right)^{t/30} = \left(\frac{1}{2}\right)^5$$

$\frac{t}{30} = 5 \Rightarrow$ t = 150

Thus, 150 hours have passed since t = 0.

Lesson 14 Supplementary Exercises

In Exercises 1-10, solve for x.

1. $169^x = 13^x$
2. $7^x = \frac{1}{49}$
3. $\left(\frac{3}{5}\right)^{x+1} = \frac{25}{9}$
4. $4^{3x+2} < 64$
5. $\left(\frac{49}{81}\right)^{x+1} \geq \frac{9}{7}$
6. $4^{x+1} = \frac{1}{64}$
7. $5^{3x+8} = 25^{2x}$
8. $4^{5x-13} = \frac{1}{8^x}$
9. $10^x > 100^{-2x-5}$
10. $49^x = 343^{2x-3}$

11. How much time is needed for a sample of Pd-100 to lose 93.75% of its original amount? Pd-100 has a half-life of 3.634 days.[3]

12. A researcher is investigating a specimen of bacteria. She finds that the original 1000 bacteria grew to 2,048,000 in 60 hours. How fast does the bacteria **(a)** double? **(b)** quadruple?

[3] http://www.periodictable.com/Isotopes/046.100/index.html

Lesson 15: Graphing Exponential Functions

Learning Outcome(s): At the end of the lesson, the learner is able to represent an exponential function through its (a) table of values, (b) graph, and (c) equation, find the domain and range of an exponential function, determine the intercepts, zeroes, and asymptotes of an exponential function, and graph exponential functions

Lesson Outline:

1. Graphs of $f(x) = b^x$ for $b > 1$ and for $0 < b < 1$
2. Domain, range, intercepts, zeroes, and asymptotes.

In the following examples, the graph is obtained by first plotting a few points. Results will be generalized later on.

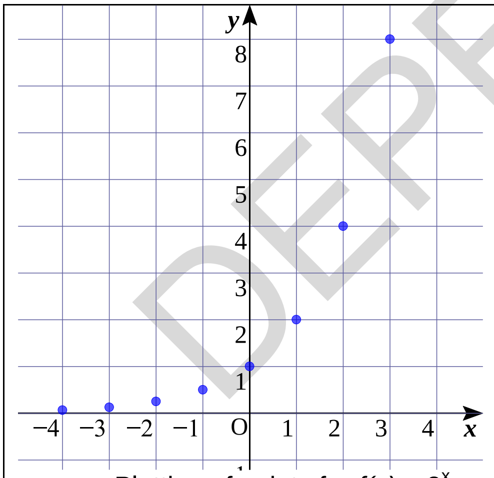
Example 1. Sketch the graph of $f(x) = 2^x$.

Solution.

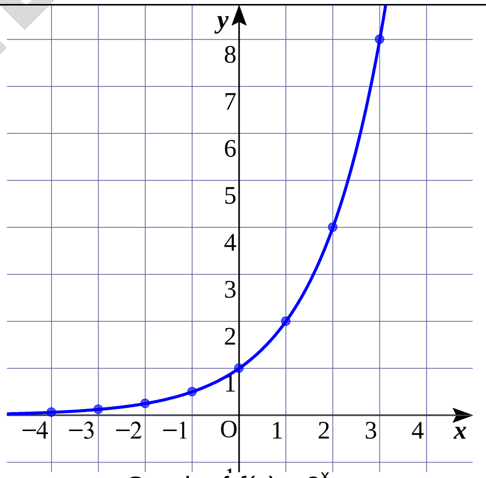
Step 1: Construct a table of values of ordered pairs for the given function. The table of values for $f(x) = 2^x$ is as follows:

x	-4	-3	-2	-1	0	1	2	3
f(x)	1/16	1/8	1/4	1/2	1	2	4	8

Step 2: Plot the points found in the table, and connect them using a smooth curve.



Plotting of points for *f(x)* = 2^x



Graph of *f(x)* = 2^x

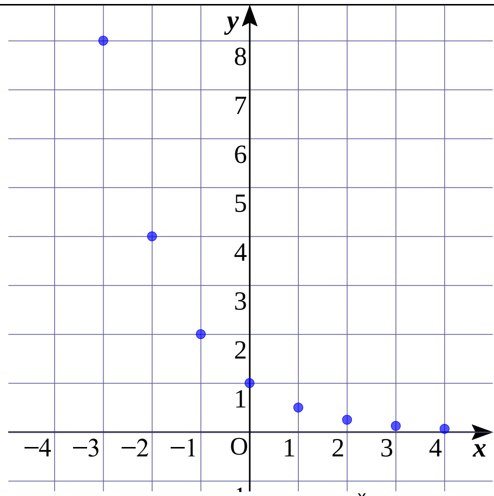
It can be observed that the function is defined for all values of x, is strictly increasing, and attains only positive y-values. As x decreases without bound, the function approaches 0, i.e., the line $y = 0$ is a horizontal asymptote.

Example 2. $g(x) = (1/2)^x$

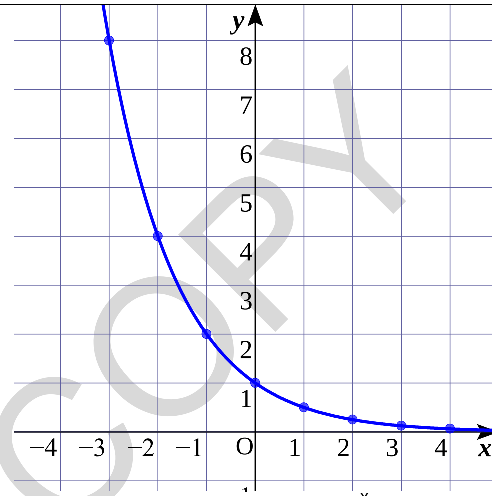
Step 1: The corresponding table of values of x and y for $g(x) = (1/2)^x$ is as follows:

x	-3	-2	-1	0	1	2	3	4
g(x)	8	4	2	1	1/2	1/4	1/8	1/16

Step 2: Plot the points found in the table and connect them using a smooth curve.

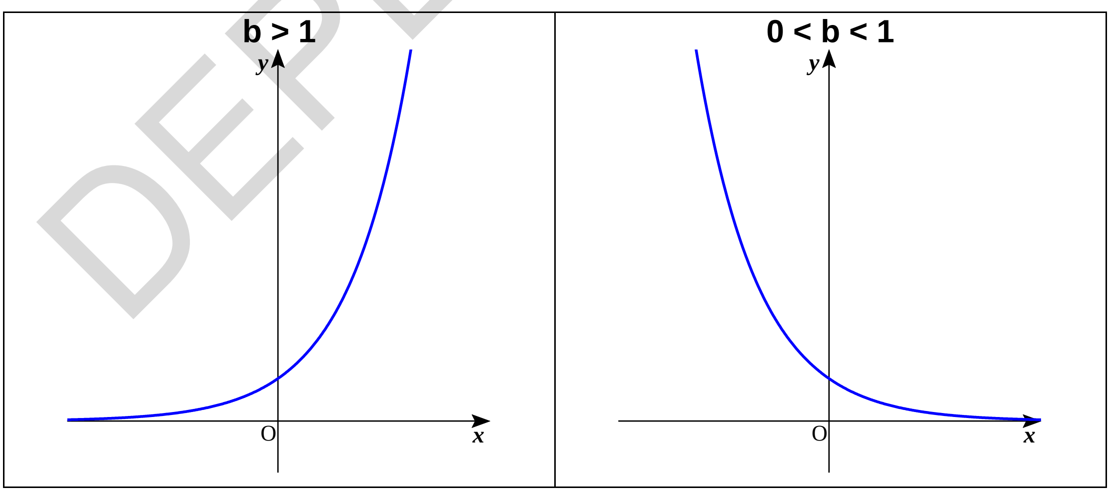


Plotting of points for $g(x) = (1/2)^x$



Graph of $g(x) = (1/2)^x$

It can be observed that the function is defined for all values of x, is strictly decreasing, and attains only positive values. As x increases without bound, the function approaches 0, i.e., the line y = 0 is a horizontal asymptote. In general, depending on the value of b, the graph of $f(x) = b^x$has the following grap



PROPERTIES OF EXPONENTIAL FUNCTIONS:

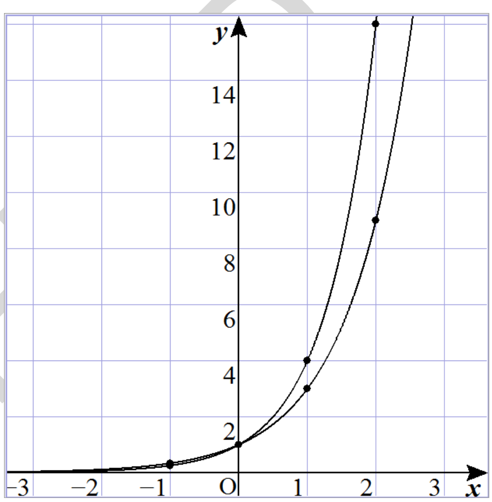
1. The domain is the set of all real numbers.
2. The range is the set of all positive real numbers.
3. It is a one-to-one function. It satisfies the Horizontal Line Test.
4. The y-intercept is 1. There is no x-intercept.
5. The horizontal asymptote is the line y = 0 (or the x-axis). There is no vertical asymptote.
6. The function is increasing if b > 1, and is decreasing if 0 < b < 1.

Solved Examples

1. Graph the functions $f(x) = 3^x$ and $g(x) = 4^x$ in the same coordinate plane. Indicate the domain, range, y-intercept, and horizontal asymptote. Compare the two graphs.

Solution. For both these functions, the base is greater than 1. Thus, both functions are increasing. The following table of values will help complete the sketch.

x	-2	-1	0	1
f(x)	1/9	1/3	1	3
g(x)	1/16	1/4	1	4



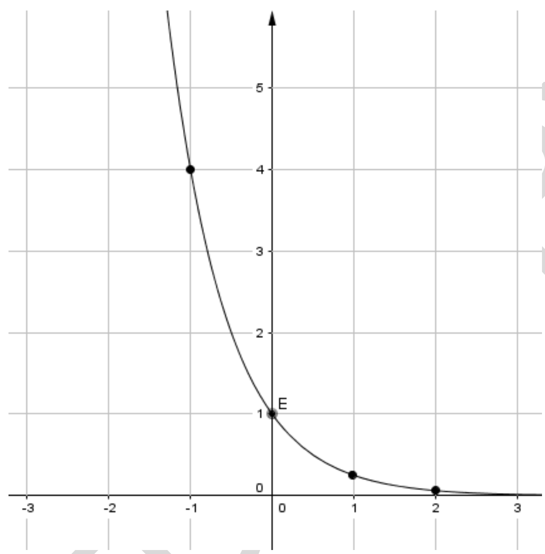
<u>For both functions:</u>
Domain: Set of all real numbers
Range: Set of all positive real numbers
y-intercept: 1. There is no x-intercept
Horizontal Asymptote: y = 0

The two graphs have the same domain, range, y-intercept, and horizontal asymptote. However, the graph of $g(x) = 4^x$ rises faster than does $f(x) = 3^x$ as x increases, and is closer to the x-axis if x < 0.

2. $g(x) = 4^{-x}$

Solution. The function $g(x) = 4^{-x}$ can be written as $g(x) = \left(\frac{1}{4}\right)^x$. This is an exponential function with base b < 1. Thus, the function is decreasing. The following table of values will help complete the sketch.

x	-1	0	1	2
f(x)	4	1	1/4	1/16



Domain: Set of all real numbers
Range: Set of all positive real numbers
y-intercept: 1. There is no x-intercept
Horizontal Asymptote: y = 0

Lesson 15 Supplementary Exercises

Construct a table of values for the given functions below using **(a)** the values of x = -2, -1, 0, 1, 2. **(b)** Sketch their graphs on a coordinate plane. **(c)** For each function, label the domain, range, y-intercept and horizontal asymptote.

1. $f(x) = \left(\frac{1}{5}\right)^x$ 2. $f(x) = 6^x$ 3. $f(x) = 3^{-x}$

Lesson 16: Graphing Transformations of Exponential Functions

Learning Outcome(s): At the end of the lesson, the learner is able to graph exponential functions.

Lesson Outline:

1. Vertical and horizontal reflection
2. Stretching and shrinking
3. Vertical and horizontal shifts

Reflecting Graphs

Example 1. Use the graph of $y = 2^x$ to graph the functions $y = -2^x$ and $y = 2^{-x}$.

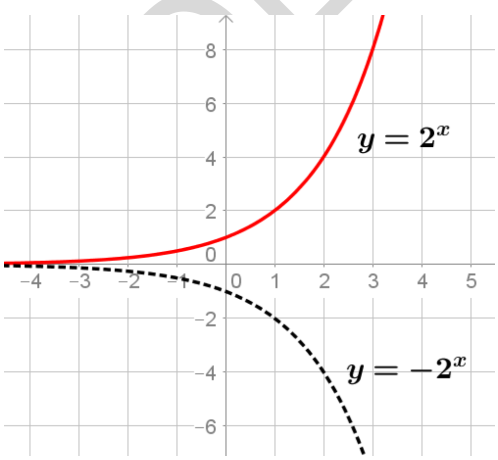
Solution. Some y-values are shown on the following table.

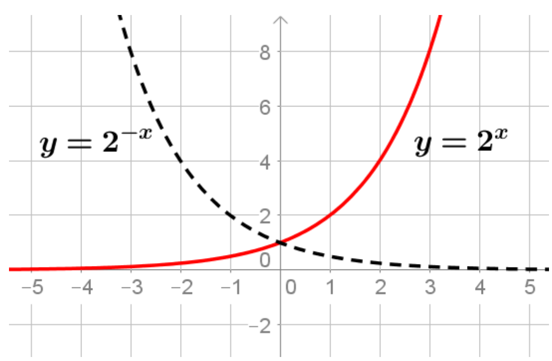
x	–3	–2	–1	0	1	2	3
$y = 2^x$	0.125	0.25	0.5	1	2	4	8
$y = -2^x$	–0.125	–0.25	–0.5	–1	–2	–4	–8
$y = 2^{-x}$	8	4	2	1	0.5	0.25	0.125

The y-coordinate of each point on the graph of $y = -2^x$ the negative of the y-coordinate of the graph of $y = 2^x$. Thus, the graph of $y = -2^x$ is the reflection of the graph of $y = 2^x$ about the x-axis.

The value of $y = 2^{-x}$ at x is the same as the value of $y = 2^x$ at –x. Thus, the graph of $y = 2^{-x}$ is the reflection of the graph of $y = 2^x$ about the y-axis.

The corresponding graphs are shown below.





The results in Example 1can be generalized as follows:

Reflection

The graph of y = –f(x) is the reflection about the x-axis of the graph of y = f(x).

The graph of y = f(–x) is the reflection about the y-axis of the graph of y = f(x).

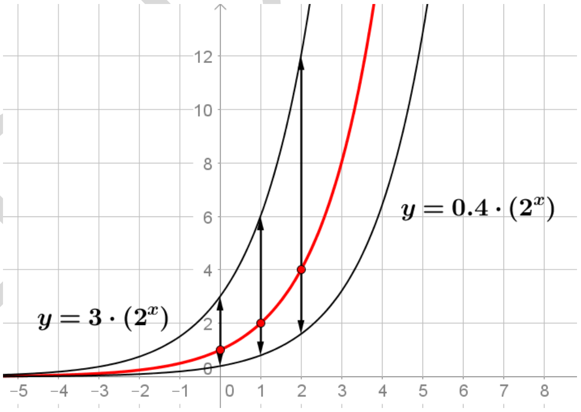
Example 2. Use the graph of $y = 2^x$ to graph the functions $y = 3(2^x)$ and $y = 0.4(2^x)$.

Solution. Some y-values are shown on the following table.

x	–3	–2	–1	0	1	2	3
$y = 2^x$	0.125	0.25	0.5	1	2	4	8
$y = 3(2^x)$	0.375	0.75	1.5	3	6	12	24
$y = 0.4(2^x)$	0.05	0.1	0.2	0.4	0.8	1.6	3.2

The y-coordinate of each point on the graph of $y = 3(2^x)$ is 3 times the y-coordinate of each point on $y = 2^x$. Similarly, the y-coordinate of each point on the graph of $y = 0.4(2^x)$ is 0.4 times the y-coordinate of each point on $y = 2^x$.

The graphs of these functions are shown below.



Observations.

1. The domain for all three graphs is the set of all real numbers.

2. The y-intercepts were also multiplied correspondingly. The y-intercept of $y = 3(2^x)$ is 3, and the y-intercept of $y = 0.4(2^x)$ is 0.4.

3. All three graphs have the same horizontal asymptote: y = 0.

4. The range of all three graphs is the set of all y > 0.

The results of Example 2 can be generalized as follows.

Vertical Stretching or Shrinking

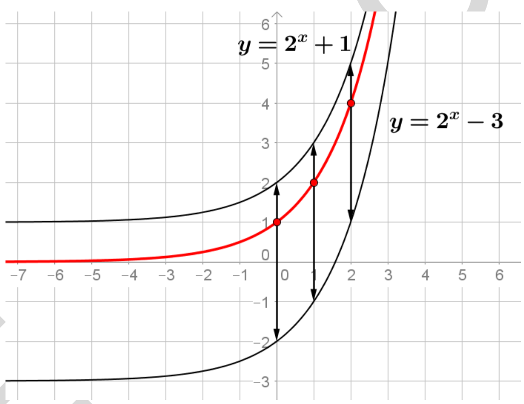
Let c be a positive constant. The graph of y = cf(x) can be obtained from the graph of y = f(x) by multiplying each y-coordinate by c. The effect is a vertical stretching (if c > 1) or shrinking (if c < 1) of the graph of y = f(x).

Example 3. Use the graph of $y = 2^x$ to graph $y = 2^x – 3$ and $y = 2^x + 1$.

***Solution*.** Some y-values are shown on the following table.

x	–3	–2	–1	0	1	2	3
$y = 2^x$	0.125	0.25	0.5	1	2	4	8
$y = 2^x – 3$	–2.875	–2.75	–2.5	–2	–1	1	5
$y = 2^x + 1$	1.125	1.25	1.5	2	3	5	9

The graphs of these functions are shown below.



Observations.

1. The domain for all three graphs is the set of all real numbers.

2. The y-intercepts and horizontal asymptotes were also vertically translated from the y-intercept and horizontal asymptote of $y = 2^x$.

3. The horizontal asymptote of $y = 2^x$ is y = 0. Shift this 1 unit up to get the horizontal asymptote of $y = 2^x + 1$ which is y = 1, and 3 units down to get the horizontal asymptote of $y = 2^x – 3$ which is y = – 3.

4. The range of $y = 2^x + 1$ is all y > 1, and the range of $y = 2^x – 3$ is all y > –3.

The results of Example 3 can be generalized as follows.

Vertical Shifts

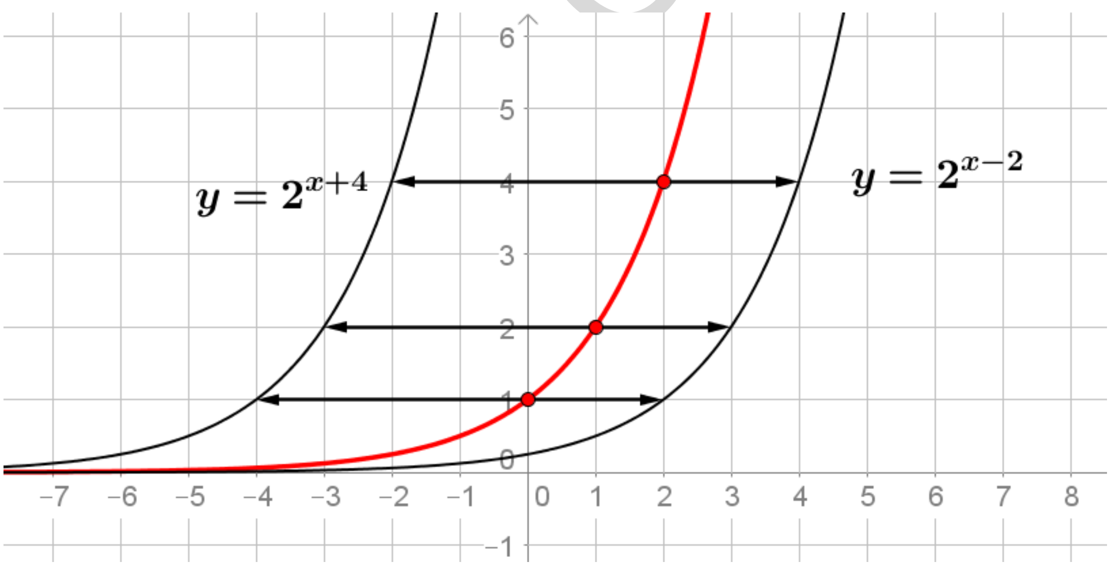
Let k be a real number. The graph of $y = f(x) + k$ is a vertical shift of k units up (if $k > 0$) or k units down (if $k < 0$) of the graph of $y = f(x)$.

Example 4. Use the graph of $y = 2^x$ to graph $y = 2^{x-2}$ and $y = 2^{x+4}$

Solution. Some y-values are shown on the following table.

x	–3	–2	–1	0	1	2	3
$y = 2^x$	0.125	0.25	0.5	1	2	4	8
$y = 2^{x-2}$	0.031	0.063	0.125	0.25	0.5	1	2
$y = 2^{x+4}$	2	4	8	16	32	64	128

The graphs of these functions are shown below.



Observations. 1. The domain for all three graphs is the set of all real numbers.

2. The y-intercepts changed. To find them, substitute x = 0 in the function. Thus, the y-intercept of $y = 2^{x+4}$ is $2^4 = 16$ and the y-intercept of $y = 2^{x-2}$ is $2^{-2} = .25$.

3. The horizontal asymptotes of all three graphs are the same (y = 0). Translating a graph horizontally does not change the horizontal asymptote.

4. The range of all three graphs is the set of all $y > 0$.

The results of Example 4 can be generalized as follows.

Horizontal Shifts

Let k be a real number. The graph of $y = f(x - k)$ is a horizontal shift of k units to the right (if $k > 0$) or k units to the left (if $k < 0$) of the graph of $y = f(x)$.

Solved Examples

1. Sketch the graph of $F(x) = 3^{x+1} - 2$, then state the domain, range, y-intercept, and horizontal asymptote.

Solution.

Transformation: The base function $f(x) = 3^x$ will be shifted 1 unit to the left and 2 units down.

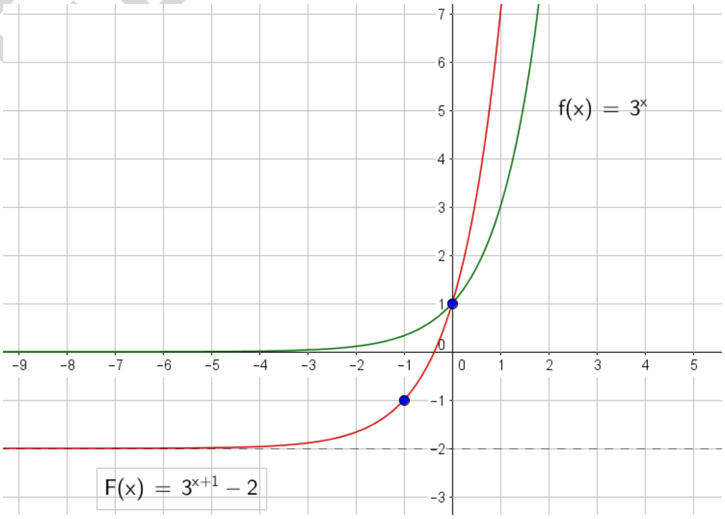
Steps in Graph Sketching:
Step 1: Base function: $f(x) = 3^x$; y-intercept: (0,1) ; horizontal asymptote: $y = 0$
Step 2: The graph of F(x) is found by shifting the graph of the function f left one unit and down two units.
Step 3: The y-intercept of f(x) (0,1) will shift to the left by one unit and down two units towards (–1, –1). This is not the y-intercept of F(x).
Step 4: The horizontal asymptote will be shifted down two units, which is $y = -2$.
Step 5: Find additional points on the graph; $F(0) = 3^{0+1} - 2 = 1$ and $F(1) = 3^{1+1} - 2 = 7$. [(0,1) and (1,7)]
Step 6: Connect the points using a smooth curve.

Domain: All real numbers

Range: $(-2, \infty)$

y-intercept: (0,1)

Horizontal Asymptote: $y = -2$



2. Sketch the graph of $G(x) = 4\left(\frac{1}{2}\right)^x + 1$, then state the domain, range, y-intercept, and horizontal asymptote.

Solution.

Transformation: The base function g(x) = $(1/2)^x$ will be stretched 4 units (that is, every y-value will be multiplied by 4), then will be shifted 1 unit upward.

Steps in Graph Sketching:

Step 1: Base function: g(x) = $\left(\frac{1}{2}\right)^x$; y-intercept: (0,1) ; horizontal asymptote: y = 0

Step 2: The graph of G(x) is obtained by stretching the graph of g by four units then shifting the graph upward by one unit.

Step 3: Since the graph will be stretched by 4 units, the y-intercept of g(x) (0,1) will be at (0,4), then will be shifted again by 1 unit upward to get (0, 5). This is the y-intercept of G(x).

Step 4: The horizontal asymptote will be shifted 1 unit upward, which is y = 1.

Step 5: Find additional points in the graph: G(-1) = $4(1/2)^{-1} + 1 = 9$ and G(3) = $4(1/2)^3 + 1 = 3/2$. [(–1,1/2) and (3, 3/2)]

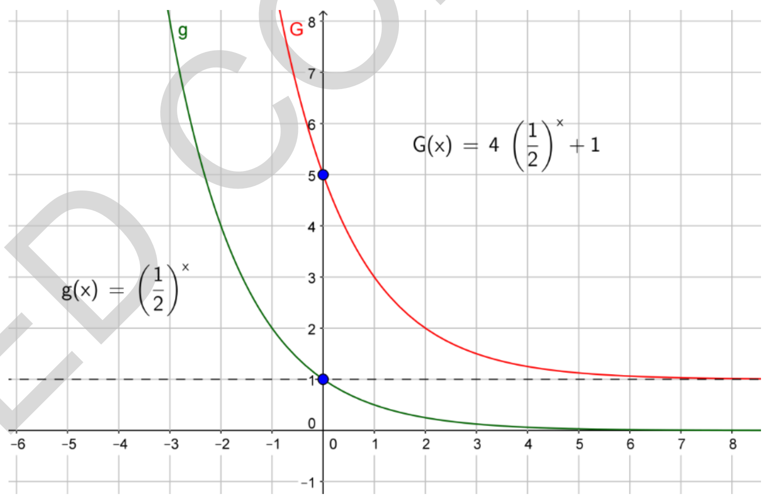
Step 6: Connect the points using a smooth curve.

Domain: All real numbers

Range: (1, ∞)

y-intercept: (0,5)

Horizontal Asymptote: y = 1



Lesson 16 Supplementary Exercises

In Exercises 1-3, **(a)** use transformations to describe how the graph is related to its base exponential function y = b^x, **(b)** sketch the graph, **(c)** identify its domain, range, y-intercept, and horizontal asymptote.

1. F(x) = $2 \cdot 3^x$ 2. G(x) = $(1/4)^{x+1} - 4$ 3. H(x) = $-2(3^{x-1})$

4. Find an exponential function of the form f(x) = $a \cdot b^x + c$ such that the y-intercept is –5, the horizontal asymptote is y = –10, and f(2) = 35.

5. Give the range of the function y = 3^x for $-10 \le x \le 10$.

Lessons 12 – 16 Topic Test 1

1. Solve for x. **[10]**

a. $3^{-x} = 27^{x+2}$ b. $\left(\frac{2}{5}\right)^{x-2} = \left(\frac{25}{4}\right)^{3x}$

2. Solve the inequality $5^x > \frac{1}{125}$. **[10]**

3. The population of a certain city doubles every 50 years. **[15]**

   (a) Give an exponential model for this situation.

   (b) By what factor does the population increase after 30 years?

   (c) If the city's population is currently 100,000, how long will it take for the population to exceed 400,000?

4. Graph the following functions. Label all intercepts and asymptotes. Indicate the domain and range. **[15]**

a. $f(x) = 3^{x-2} + 1$ b. $h(x) = (0.1)^{x+2} - 1$

Lessons 12 – 16 Topic Test 2

1. Solve for x in the following equations. **[10]**

a. $\frac{4}{9} = \left(\frac{2}{3}\right)^{x-1}$ b. $8^{1-2x} = 16^{2x-1}$

2. Solve the inequality $\left(\frac{1}{3}\right)^{2x} \leq \left(\frac{1}{27}\right)^{1-x}$ **[10]**

3. Maine decides to participate in an investment that yields 3.75% interest annually. If she invests ₱12,500, how much will her investment be after 5 years? **[10]**

4. The population of a certain bacteria colony is modeled by the function $P(x) = 500e^{0.05t}$, where P(x) is the bacteria population after t minutes. Find the bacteria population after half an hour. **[10]**

5. Graph the function $f(x) = 2(1/2)^{x-1}$. Label all intercepts and asymptotes. Indicate the domain and range. **[10]**

Lesson 17: Introduction to Logarithms

Learning Outcome(s): At the end of the lesson, the learner is able to represent real-life situations using logarithmic functions and solve problems involving logarithmic functions.

Lesson Outline:

1. Logarithms, including common and natural logarithms
2. Applications (Richter scale, decibels, pH levels)

Definition: Let a, b, and c be positive real numbers such that $b \neq 1$. The **logarithm** of a with base b is denoted by $\log_b a$, and is defined as

$$c = \log_b a \text{ if and only if } a = b^c$$

Reminders.

1. In both the logarithmic and exponential forms, b is the base. In the exponential form, c is an exponent; this implies that the logarithm is actually an exponent. Hence, logarithmic and exponential functions are inverses.

2. In the logarithmic form $\log_b x$, x cannot be negative.

3. The value of $\log_b x$ can be negative.

Definition: Common logarithms are logarithms with base 10; logx is a short notation for $\log_{10} x$.

Definition: Natural logarithms are logarithms to the base e (approximately 2.71828), and are denoted by "ln". In other words, lnx is another way of writing $\log_e x$.

Example 1. Rewrite the following exponential equations in logarithmic form, whenever possible.

a. $5^3 = 125$ b. $7^{-2} = 1/49$ c. $10^2 = 100$ d. $(2/3)^2 = 4/9$ e. $(0.1)^{-4} = 10000$

f. $4^0 = 1$ g. $7^b = 21$ h. $e^2 = x$ i. $(-2)^2 = 4$

Solution.

a. $\log_5 125 = 3$ b. $\log_7(1/49) = -2$ c. $\log 100 = 2$ d. $\log_{2/3}(4/9) = 2$

e. $\log_{0.1} 10000 = -4$ f. $\log_4 1 = 0$ g. $\log_7 21 = b$ h. $\ln x = 2$

i. cannot be written in logarithmic form

Example 2. Rewrite the following logarithmic equations in exponential form.

a. $\log m = n$ b. $\log_3 81 = 4$ c. $\log_{\sqrt{5}} 5 = 2$ d. $\log_{3/4}(64/27) = -3$
e. $\log_4 2 = ½$ f. $\log_{10} 0.001 = -3$ g. $\ln 8 = a$

Solution.

a. $10^n = m$ b. $3^4 = 81$ c. $\left(\sqrt{5}\right)^2 = 5$ d. $(3/4)^{-3} = 64/27$
e. $4^{1/2} = 2$ f. $10^{-3} = 0.001$ g. $e^a = 8$

Example 3. Find the value of the following logarithmic expressions.

a. $\log_2 32$ b. $\log_9 729$ c. $\log 0.001$ d. $\log_{1/2} 16$
e. $\log_7 1$ f. $\log_5 5$

Solution.
a. 5   b. 3   c. –3   d. –4   e. 0   f. 1

Applications. Some of the most common applications in real-life of logarithms are the Richter scale, sound intensity, and pH level.

In 1935, Charles Richter proposed a logarithmic scale to measure the intensity of an earthquake. He defined the magnitude of an earthquake as a function of its amplitude on a standard seismograph. The following formula produces the same results, but is based on the energy released by an earthquake.[4]

Earthquake Magnitude on a Richter scale

The magnitude *R* of an earthquake is given by

$$R = \frac{2}{3}\log\frac{E}{10^{4.40}}$$

where *E* (in joules) is the energy released by the earthquake (the quantity $10^{4.40}$ joules is the energy released by a very small reference earthquake).

The formula indicates that the magnitude of an earthquake is based on the logarithm of the ratio between the energy it releases and the energy released by the reference earthquake.

Example 4. Suppose that an earthquake released approximately 10^{12} joules of energy. **(a)** What is its magnitude on a Richter scale? **(b)** How much more energy does this earthquake release than the reference earthquake?

Solution.

(a) Since $E = 10^{12}$, then $R = \frac{2}{3}\log\frac{10^{12}}{10^{4.40}} = \frac{2}{3}\log 10^{7.6}$.
Since by definition, $\log 10^{7.6}$ is the exponent by which 10 must be raised to obtain $10^{7.6}$, then $\log 10^{7.6} = 7.6$.
Thus, $R = \frac{2}{3}(7.6) \approx 5.1$.

[4]Barnett, R. A., Ziegler, M. R., Byleen, K. E., & Sobecki, D. (2008). *Precalculus.* New York: McGraw-Hill.

(b) This earthquake releases $10^{12}/10^{4.40} = 10^{7.6} \approx 39810717$ times more energy than the reference earthquake.

Sound Intensity[5]

In acoustics, the decibel (dB) level of a sound is

$$D = 10\log\frac{I}{10^{-12}}$$

where I is the sound intensity in watts/m^2 (the quantity 10^{-12} watts/m^2 is least audible sound a human can hear).

Example 5. The decibel level of sound in a quiet office is 10^{-6}watts/m^2. (a) What is the corresponding sound intensity in decibels? (b) How much more intense is this sound than the least audible sound a human can hear?

Solution.

(a) $D = 10\log\frac{10^{-6}}{10^{-12}} = 10\log 10^6$.Since by definition, log 10^6 is the exponent by which 10 must be raised to obtain 10^6, then log $10^6 = 6$.

Thus, D = 10(6) = 60 decibels.

(b) This sound is $10^{-6}/10^{-12} = 10^6 = 1{,}000{,}000$ times more intense than the least audible sound a human can hear.

Acidity and the pH scale[6]

The pH level of a water-based solution is defined as

$$\text{pH} = -\log[H^+],$$

where [H+] is the concentration of hydrogen ions in moles per liter. Solutions with a pH of 7 are defined **neutral**; those with pH < 7 are **acidic**, and those with pH > 7 are **basic**.

Example 6. A 1-liter solution contains 0.00001 moles of hydrogen ions. Find its pH level.

Solution. Since there are 0.00001 moles of hydrogen ions in 1 liter, then the concentration of hydrogen ions is 10^{-5} moles per liter.

The pH level is $-\log 10^{-5}$. Since log 10^{-5} is the exponent by which 10 must be raised to obtain 10^{-5}, then log $10^{-5} = -5$.

Thus, pH = $-\log 10^{-5} = -(-5) = 5$.

[5] Young, C. (2012). *College algebra* (3rd ed). Hoboken, NJ: John Wiley & Sons.

[6] Stewart, J., Redlin, L., & Watson, S. (2012). *Precalculus: Mathematics for calculus* (6th ed). Belmont, CA: Brooks/Cole, Cengage Learning.

Solved Examples

In numbers 1-3, find the value of the following logarithmic expressions.

1. $\log_3 81$ 2. $\log_{169} 13$ 3. $\log_5\left(\frac{1}{5}\right)$

Answers. 1. 4 2. 2 3. –1

In numbers 4-6, rewrite the following expressions in logarithmic form, whenever possible.

4. $16 = 2^4$ 5. $9 = \sqrt{81}$ 6. $\frac{1}{9} = 3^{-2}$

Answers. 4. $\log_2 16 = 4$ 5. $\log_{81} 9 = \frac{1}{2}$ 6. $\log_3 \frac{1}{9} = -2$

In numbers 7-9, rewrite the following logarithmic equations in exponential form, whenever possible.

7. $\log_3 9 = 2$ 8. $\log_{16} 4 = \frac{1}{2}$ 9. ln x =1

Answers. 7. $3^2 = 9$ 8. $16^{1/2} = 4$ 9. $e^1 = x$

10. What is the magnitude in the Richter scale of an earthquake that released 10^{14} joules of energy? How much more energy does this earthquake release than that of the reference earthquake?

Answer. Magnitude $R = \frac{2}{3}\log\frac{10^{14}}{10^{4.4}} = \frac{2}{3}\log 10^{9.6} = \frac{2}{3}(9.6) = 6.4$. The earthquake released $10^{14}/10^{4.40} = 10^{9.6} = 3981071706$ times more energy than that by the reference earthquake.

11. Suppose the intensity of sound of a jet during takeoff is 100 watts/m^2. What is the corresponding sound intensity in decibels? How much more intense is this sound than the least audible sound a human can hear?

Answer. Sound intensity $D = 10\log\frac{100}{10^{-12}} = 10\log\frac{10^2}{10^{-12}} = 10\log 10^{14} = 10(14) =$ 140 decibels. This sound is $10^2/10^{-12} = 10^{14}$ times more intense than the least audible sound a human can hear.

Lesson 17 Supplementary Exercises

In numbers 1-3, find the value of the following logarithmic expressions.

1. $\log_2\left(\frac{1}{2}\right)$ 2. $\log_{100} 10$ 3. $\log_{0.5} 4$

In numbers 4-6, rewrite the following exponential equations in logarithmic form.

4. $81 = 9^2$ 5.$12 = \sqrt{144}$ 6. $\frac{1}{49} = 7^{-2}$

In numbers 7-9, rewrite the following logarithmic equations in exponential form.

7.$\log_2 8 = 3$ 8.$\log_9 3 = \frac{1}{2}$ 9.$\log_5\left(\frac{1}{25}\right) = -2$

10. What is the magnitude in the Richter scale of an earthquake that released 10^{16} joules of energy?

11. A 1-liter solution contains 10^{-8} moles of hydrogen ions. Determine whether the solution is acidic, neutral, or basic.

Lesson 18: Logarithmic Functions, Equations, and Inequalities

Learning Outcome(s): At the end of the lesson, the learner is able to distinguish among logarithmic function, logarithmic equation, and logarithmic inequality.

Lesson Outline:

1. Logarithmic equations, logarithmic inequalities, and logarithmic functions

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The definitions of exponential equations, inequalities and functions are shown below.

	Logarithmic Equation	Logarithmic Inequality	Logarithmic Function
Definition	An equation involving logarithms.	An inequality involving logarithms.	Function of the form $f(x) = \log_b x$ $(b > 0, b \neq 1)$.
Example	$\log_x 2 = 4$	$\ln x^2 > (\ln x)^2$	$g(x) = \log_3 x$

A logarithmic equation or inequality can be solved for all x values that satisfy the equation or inequality (Lesson 21). A logarithmic function expresses a relationship between two variables (such as x and y), and can be represented by a table of values or a graph (Lesson 22).

Solved Examples

Determine whether the given is a logarithmic function, a logarithmic equation, a logarithmic inequality or neither.

1. $g(x) = \log_5 x$ (Answer: Logarithmic Function)
2. $y = 2\log_4 x$ (Answer: Logarithmic Function)
3. $\log(4x) = -\log(3x + 5)$ (Answer: Logarithmic Equation)
4. $x\log_2(x) - 1 > 0$ (Answer: Logarithmic Inequality)
5. $\log x(x - 3) = \log 4$ (Answer: Logarithmic Equation)

Lesson 18 Supplementary Exercises

Determine whether the given is a logarithmic function, a logarithmic equation, a logarithmic inequality or neither.

1. $\log_3(2x - 1) > \log_3 x + 2$
2. $h(x) = \log_{0.25} x$
3. $2 + y = \log_3 x$
4. $\log_3(2x - 1) = 2$
5. $\log x^2 = 2$

Lesson 19: Basic Properties of Logarithms

Learning Outcome(s): At the end of the lesson, the learner is able to apply basic properties of logarithms and solve problems involving logarithmic equations

Lesson Outline:

1. Basic properties of logarithms.
2. Simplifying logarithmic expressions.

Definition: Let b and x be real numbers such that $b > 0$ and $b \neq 1$, the **basic properties of logarithms** are as follows:

1. $\log_b 1 = 0$
2. $\log_b b^x = x$
3. If $x > 0$, then $b^{\log_b x} = x$

Example 1. Use the basic properties of logarithms to find the value of the following logarithmic expressions.

a. $\log 10$ b. $\ln e^3$ c. $\log_4 64$ d. $\log_5(1/125)$ e. $5^{\log_5 2}$ f. $\log 1$

Solution.

a. $\log 10 = \log_{10} 10^1 = 1$ (Property 2)

b. $\ln e^3 = \log_e e^3 = 3$ (Property 2)

c. $\log_4 64 = \log_4 4^3 = 3$ (Property 2)

d. $\log_5(1/125) = \log_5 5^{-3} = -3$ (Property 2)

e. $5^{\log_5 2} = 2$ (Property 3)

f. $\log 1 = 0$ (Property 1)

EXAMPLE 2. Suppose you have seats to a concert featuring your favorite musical artist. Calculate the approximate decibel level associated if a typical concert's sound intensity is 10^{-2} W/m^2. (Refer to Lesson 17 for a discussion of sound intensity).

Solution.

$D = 10\log\left(\frac{I}{I_0}\right)$

$D = 10\log\left(\frac{10^{-2}}{10^{-12}}\right)$

$D = 10\log(10^{10})$

$D = 10 \cdot 10$ (Property 2)

$D = 100$ dB

Answer. A concert's decibel level is 100dB.

EXAMPLE 3. Calculate the hydrogen ion concentration of vinegar that has a pH level of 3.0. (Refer to Lesson 17 for a discussion of pH levels).

Solution.

pH = –log[H^+]

3.0 = –log[H^+]

–3.0 = log[H^+]

$10^{-3.0} = 10^{\log H^+}$

$10^{-3.0}$ = [H^+] (Property 3)

Answer. The hydrogen ion concentration is $10^{-3.0}$ moles per liter.

Solved Examples

1. Use the basic properties of logarithms to find the value of the following logarithmic expressions:

a. $\log_7 7$ b. $e^{\ln 5}$ c. $\log 10^{x^2}$ d. $\log_3 1$ e. $\ln e^{(x+1)}$ f. $\log_7 \left(\frac{1}{49}\right)$

Solution.

a. $\log_7 7 = \log_7 7^1 = 1$

b. $e^{\ln 5} = 5$ (Property 3)

c. $\log 10^{x^2} = \log_{10} 10^{x^2} = x^2$

d. $\log_3 1 = 0$ (Property 1)

e. $\ln e^{(x+1)} = \log_e e^{(x+1)} = x + 1$

f. $\log_7 \left(\frac{1}{49}\right) = \log_7 7^{-2} = -2$

2. Calculate the sound intensity in watts/m^2 of a 65-decibel sound. (Refer to Lesson 17 for a discussion of sound intensity).

Solution.

$D = 10 \log \frac{I}{10^{-12}}$.

Solve $65 = 10 \log \frac{I}{10^{-12}}$.

$6.5 = \log \frac{I}{10^{-12}}$

$10^{6.5} = 10^{\log \frac{I}{10^{-12}}}$

$10^{6.5} = \frac{I}{10^{-12}}$ (Property 3)

$I = 10^{6.5} 10^{-12} \approx 3.16 \times 10^{-6}$ watts/m^2.

Lesson 19 Supplementary Exercises

In numbers 1-5, use the three basic properties to find the value of the logarithm expressions below:

1. $\log_2 2^{-2}$ 2. $\ln e^{-5}$ 3. $\log_{100} 1$ 4. $\log 10^{(-x+1)}$ 5. $e^{\ln 3}$

6. To measure the brightness of a star from earth, the brightness of the star Vega is used as a reference, and is assigned a relative intensity $I_0 = 1$. The magnitude m of any given star is defined by m = 2.5 log I, where I is the relative intensity of that star.[7] (a) What is the magnitude of Vega? (b) Suppose that light arriving from another star has a relative intensity of 5.32. What is the magnitude of this star?

Lesson 20: Laws of Logarithms

Learning Outcome(s): At the end of the lesson, the learner is able to illustrate the laws of logarithms.

Lesson Outline:

1. Laws of logarithms
2. Change of base formula

Laws of Logarithms. Let b > 0, b ≠ 1 and let $n \in \mathbb{R}$. For u > 0, v > 0, then

1. $\log_b(uv) = \log_b u + \log_b v$ (Example: $\log_2(3x) = \log_2 3 + \log_2 x$)
2. $\log_b(u/v) = \log_b u - \log_b v$ (Example: $\log_3(4/5) = \log_3 4 - \log_3 5$)
3. $\log_b u^n = n \cdot \log_b u$ (Example: $\log_5 36 = \log_5 6^2 = 2\log_5 6$)

Example 1. Use the properties of logarithms to expand each expression in terms of the logarithms of the factors. Assume each factor is positive.

a. $\log(ab^2)$

Solution.

$$\log(ab^2) = \log a + \log b^2$$
$$= \log a + 2\log b$$

b. $\log_3(3/x)^3$

Solution.

$$\log_3(3/x)^3 = 3\log_3(3/x)$$
$$= 3(\log_3 3 - \log_3 x)$$
$$= 3(1 - \log_3 x) = 3 - 3\log_3 x$$

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[7] Crauder, B., Evans, B., & Noell, A. (2008). *Functions and change: A modeling approach to college algebra and trigonometry.* Boston: Houghton Mifflin.

c. $\ln[x(x-5)]$

Solution.

$$\ln[x(x-5)] = \ln x + \ln(x-5)$$

Example 2. Use the properties of logarithm to condense the expressions as a single logarithm.

a. $\log 2 + \log 3$

Solution.

$$\log 2 + \log 3 = \log(2\cdot 3)$$
$$= \log 6$$

b. $2\ln x - \ln y$

Solution.

$$2\ln x - \ln y = \ln x^2 - \ln y$$
$$= \ln(x^2/y)$$

c. $\log_5(x^2) - 3\log_5 x$

Solution.

$$\log_5(x^2) - 3\log_5 x = \log_5(x^2) - \log_5(x^3)$$
$$= \log_5(x^2/x^3)$$
$$= \log_5(1/x)$$
$$= \log_5(x^{-1})$$
$$= -\log_5 x$$

d. $2 - \log 5$

Solution.

$2 = 2(1) = 2(\log 10) = \log 10^2 = \log 100$

$$2 - \log 5 = \log 100 - \log 5$$
$$= \log(100/5)$$
$$= \log 20$$

Change-of-base formula

Any logarithmic expression can be expressed as a quotient of two logarithmic expressions with a common base. Let a, b, and x be positive real numbers, with a ≠ 1, b ≠ 1:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Example 3. Use the change-of-base formula to rewrite the following logarithmic expressions to the indicated base.

a. $\log_6 4$ (change to base 2)

Solution.

$$\log_6 4 = \frac{\log_2 4}{\log_2 6}$$

$$= \frac{2}{\log_2 6}$$

b. $\log_{1/2} 2$ (change to base e)

$$\log_{1/2} 2 = \frac{\ln 2}{\ln\left(\frac{1}{2}\right)}$$

$$= \frac{\ln 2}{\ln 1 - \ln 2}$$

$$= \frac{\ln 2}{0 - \ln 2}$$

$$= \frac{\ln 2}{-\ln 2}$$

$$= -1$$

Solved Examples

1. Use the properties of logarithms to expand the expressions as a sum, difference or multiple of logarithms.

a. $\log_b(u^2\sqrt{v})$

Solution.

$\log_b(u^2\sqrt{v})$

$= \log_b u^2 + \log_b v^{\frac{1}{2}}$

$= 2\log_b u + \frac{1}{2}\log_b v$

b. $\ln\left(\frac{x^3}{y^2}\right)$

Solution.

$\ln\left(\frac{x^3}{y^2}\right)$

$= \ln x^3 - \ln y^2$

$= 3\ln x - 2\ln y$

c. $\log[x(x+2)]$

Solution.

$\log[x(x+2)]$

$= \log x + \log(x+2)$

2. Use the properties of logarithms to condense the expressions as a single logarithm.

a. $\frac{2}{3}\ln x - \frac{1}{2}\ln y$

Solution.

$\frac{2}{3}\ln x - \frac{1}{2}\ln y$

$= \ln x^{2/3} - \ln y^{1/2}$

$= \ln\left(\frac{x^{2/3}}{y^{1/2}}\right)$

b. $3\log_b x + \log_b(2x+1)$

Solution.

$3\log_b x + \log_b(2x+1)$

$= \log_b x^3 + \log_b(2x+1)$

$= \log_b[x^3(2x+1)]$

c. $\log_3(9) + 2$

Solution.

$\log_3(9) + 2$

$= \log_3(9) + 2(\log_3 3)$

$= \log_3(9) + \log_3 3^2$

$= \log_3 81$

3. Use the change of base formula to rewrite the following logarithmic expressions to the indicated and then compute the approximate value using a calculator.

a. $\log_4 17$ (change to base 10)

Solution.

$\log_4 17$

$= \dfrac{\log 17}{\log 4}$

≈ 2.0437

b. $\log_9 \frac{1}{27}$ (change to base 3)

Solution.

$\log_9 \dfrac{1}{27}$

$= \dfrac{\log_3\left(\frac{1}{27}\right)}{\log_3 9}$

$= \dfrac{\log_3(3^{-3})}{\log_3 3^2}$

$= \dfrac{-3\log_3 3}{2\log_3 3}$

$= -\dfrac{3}{2}$

Lesson 20 Supplementary Exercises

1. Use the properties of logarithms to expand the expressions as a sum, difference or multiple of logarithms

1. $\log_b x^4 \sqrt[3]{y}$ 2. $\log\left(\frac{a^4}{b^5}\right)$ 3. $\log(x^{1/2}y^{1/3})$

2. Use the properties of logarithms to condense the expressions as a single logarithm.

4. $\frac{1}{2}\log a - 3 \log b$ 5. $\frac{1}{2}\ln(x+3) + \frac{1}{3}\ln(x+2)$ 6. $\log_4 x^2 + \log_4 x$

3. Use the change of base formula to rewrite the following logarithmic expressions to the indicated and then compute the approximate value using a calculator.

7. $\log_5 \frac{1}{2}$ (change to base 10)

8. $\log_{\sqrt{2}} 9$ (change to base 3)

9. $\log_{7.2} 2.5$ (change to base e)

Lesson 21: Solving Logarithmic Equations and Inequalities

Learning Outcome(s): At the end of the lesson, the learner is able to solve logarithmic equations and inequalities and solve problems involving logarithmic functions, equations, and inequalities

Lesson Outline:

1. Solve logarithmic equations
2. Solve logarithmic inequalities
3. Applications to problems in real-life contexts

Property of Logarithmic Equations

If $b > 1$, then the logarithmic function $y = \log_b x$ is increasing for all x. If $0 < b < 1$, then the logarithmic function $y = \log_b x$ is decreasing for all x. This means that $\log_b u = \log_b v$ if and only if $u = v$.

Techniques. Some strategies for solving logarithmic equations:

1. Rewriting to exponential form;

2. Using logarithmic properties;

3. Applying the one-to-one property of logarithmic functions;

4. The Zero Factor Property: If $ab = 0$, then $a = 0$ or $b = 0$.

Example 1. Find the value of x in the following equations.

a. $\log_4(2x) = \log_4 10$

Solution.

$\log_4(2x) = \log_4 10$

$2x = 10$ (one-to-one property)

$x = 5$

Check: 5 is a solution since $\log_4(2\cdot 5) = \log_4(10)$ is defined.

b. $\log_3(2x - 1) = 2$

Solution.

$\log_3(2x - 1) = 2$

$2x - 1 = 3^2$ (changing into exponential form)

$2x - 1 = 9$

$2x = 10$

$x = 5$

Check: 5 is a solution since $\log_3(2\cdot(5) - 1) = \log_3(9)$ is defined.

c. $\log_x 16 = 2$

Solution.

$\log_x 16 = 2$

$x^2 = 16$ (changing into exponential form)

$x^2 - 16 = 0$

$(x + 4)(x - 4) = 0$ (factorization using $a^2 - b^2 = (a + b)(a - b)$)

$x = -4, 4$

Check: 4 is a solution since $\log_4(16)$ is defined. However, –4 is not a solution since $\log_{-4}(16)$ is not defined (the base cannot be negative).

d. $\log_2(x + 1) + \log_2(x - 1) = 3$

Solution.

$\log_2[(x + 1)(x - 1)] = 3$ (using the property $\log_b u + \log_b v = \log_b(uv)$)

$(x + 1)(x - 1) = 2^3$ (note: Zero Factor Property cannot be used yet)

$x^2 - 1 = 8$

$x^2 - 9 = 0$ (multiplication of two binomials)

$(x + 3)(x - 3) = 0$ (factorization using $a^2 - b^2 = (a + b)(a - b)$)

$x = -3, 3$

Check: 3 is a solution since $\log_2(3+1)$ and $\log_2(3-1)$ are defined. However, -3 is not a solution since $\log_2(-3+1) = \log_2(-2)$ is not defined.

e. $\log x^2 = 2$

Solution A.

$\log x^2 = 2$

$x^2 = 10^2$ (changing into exponential form)

$x^2 = 100$

$x^2 - 100 = 0$

$(x + 10)(x - 10) = 0$

$x = -10, 10$

Check: Both are solutions since $\log(-10)^2$ and $\log(10)^2$ are defined.

Solution B.

$\log x^2 = 2$

$\log x^2 = \log 10^2 \rightarrow 2 = 2(1) = 2(\log 10) = \log 10^2$

$x^2 = 100$

$x^2 - 100 = 0$

$(x + 10)(x - 10) = 0$

$x = -10, 10$

Check: Both are solutions since $\log(-10)^2$ and $\log(10)^2$ are defined.

Incorrect Method. (using $\log_b u^n = n \cdot \log_b u$ immediately)

$\log x^2 = 2$

$2\log x = 2$ (This is not a valid conclusion because $\log x^2 = 2\log x$ only if $x > 0$).

f. $(\log x)^2 + 2\log x – 3 = 0$

Solution.

Let $\log x = A$

$A^2 + 2A – 3 = 0$

$(A + 3)(A – 1) = 0$

$A = –3$ or $A = 1$

$\log x = –3$ or $\log x = 1$

$x = 10–3 = 1/1000$ or $x = 10$

Check: Both are solutions since log(1/1000) and log10 are defined.

Example 2. Use logarithms to solve for the value of x in the exponential equation $2^x = 3$.

Solution.

$2^x = 3$

$\log 2^x = \log 3$ (applying the one-to-one property)

$x\log 2 = \log 3$ (applying $\log_b u^n = n\cdot\log_b u$ since 2 is positive)

$x = \log 3 / \log 2 \approx 1.58496$

Property of Logarithmic Inequalities

If $0 < b < 1$, then $x_1 < x_2$ if and only if $\log_b x_1 > \log_b x_2$.

If $b > 1$, then $x_1 < x_2$ if and only if $\log_b x_1 < \log_b x_2$.

Example 3. Solve the following logarithmic inequalities.

a. $\log_3(2x – 1) > \log_3(x + 2)$

Solution.

Step 1: Ensure that the logarithms are defined.

Then $2x - 1 > 0$ and $x + 2 > 0$ must be satisfied.

$2x - 1 > 0$ implies $x > 1/2$ and $x + 2 > 0$ implies $x > -2$.

To make both logarithms defined, then $x > 1/2$. (If $x > 1/2$, then x is surely greater than -2.)

Step 2: Ensure that the inequality is satisfied.

The base 3 is greater than 1.

Thus, since $\log_3(2x - 1) > \log_3(x + 2)$, then:

$$2x - 1 > x + 2$$

$$x > 3 \quad \text{(subtract x from both sides; add 1 to both sides)}$$

$$\therefore x > 3$$

Hence, the solution is $(3, +\infty)$.

Solution.

Step 1: Ensure that the logarithms are defined.

This means that $x > 0$.

Step 2: Ensure that the inequality is satisfied.

Rewrite -3 as a logarithm to base 1/5: $-3 = \log_{1/5}(1/5)^{-3}$

We obtain the inequality $\log_{1/5}x > \log_{1/5}(1/5)^{-3}$.

The base is $0.2 = 1/5$, which is less than 1.

Thus, since $\log_{1/5}x > \log_{1/5}(1/5)^{-3}$, then $x < (1/5)^{-3} = 125$.

Also, x should be positive (from Step 1). Thus, $0 < x < 125$.

Hence, the solution is $(0, 125)$.

c. $-2 < \log x < 2$

Solution.

Step 1: Ensure that the logarithms are defined.

This means that $x > 0$.

Step 2: Ensure that the inequality is satisfied.

Rewrite –2 and 2 as logarithms to the base 10, which are $\log 10^{-2}$ and $\log 10^{2}$ respectively, obtaining the inequality: $\log 10^{-2} < \log x < \log 10^{2}$.

We split the compound inequality into two simple inequalities:

$\log 10^{-2} < \log x$ and $\log x < \log 10^{2}$

Since the base 10 is greater than 1, simplify both inequalities as

$10^{-2} < x$ and $x < 10^{2}$

Thus obtaining 1/100 < x < 100, which automatically satisfies the condition in Step 1.

Hence, the solution is (1/100, 100).

EXAMPLE 4. The 2013 earthquake in Bohol and Cebu had a magnitude of 7.2, while the 2012 earthquake that occurred in Negros Oriental recorded a 6.7 magnitude. How much more energy was released by the 2013 Bohol/Cebu earthquake compared to that by the Negros Oriental earthquake?" (Refer to Lesson 17 for a discussion of the Richter scale).

Solution. Let E_B and E_N be the energy released by the Bohol/Cebu and Negros Oriental earthquakes, respectively. We will determine E_B/E_N.

Based on the given magnitudes, $7.2 = \frac{2}{3}\log\frac{E_B}{10^{4.4}}$ and $6.7 = \frac{2}{3}\log\frac{E_N}{10^{4.4}}$.

Solving for E_B: $7.2\left(\frac{3}{2}\right) = \log\frac{E_B}{10^{4.4}}$

$$10.8 = \log\frac{E_B}{10^{4.4}}$$

$$10^{10.8} = \frac{E_B}{10^{4.4}}$$

$$E_B = 10^{10.8}10^{4.4} = 10^{15.2}$$

Solving for E_N: $6.7\left(\frac{3}{2}\right) = \log\frac{E_N}{10^{4.4}}$

$$10.05 = \log\frac{E_N}{10^{4.4}}$$

$$10^{10.05} = \frac{E_N}{10^{4.4}}$$

$$E_B = 10^{10.05}10^{4.4} = 10^{14.45}$$

Thus, $E_B/E_N = 10^{15.2}/10^{14.45} = 10^{0.75} \approx 5.62$

The Bohol/Cebu earthquake released 5.62 times more energy than the Negros Oriental earthquake.

EXAMPLE 5. How much more severe is an earthquake with a magnitude of n on a Richter scale, compared to one with a magnitude of n + 1?

Solution. Let E_1 and E_2 be the energy released by the earthquakes with magnitude n and n+1, respectively. We will determine E_2/E_1.

Based on the given magnitudes, $n = \frac{2}{3}\log\frac{E_1}{10^{4.4}}$ and $n+1 = \frac{2}{3}\log\frac{E_2}{10^{4.4}}$.

Solving for E_1: $\frac{3}{2}n = \log\frac{E_1}{10^{4.4}}$

$$10^{3n/2} = \frac{E_1}{10^{4.4}}$$

$$E_1 = 10^{3n/2}10^{4.4} = 10^{\frac{3n}{2}+4.4}$$

Solving for E_2: $\frac{3}{2}(n+1) = \log\frac{E_2}{10^{4.4}}$

$$10^{3(n+1)/2} = \text{g}\frac{E_2}{10^{4.4}}$$

$$E_2 = 10^{3(n+1)/2}10^{4.4} = 10^{\frac{3(n+1)}{2}+4.4}$$

Thus, $E_2/E_1 = 10^{\frac{3(n+1)}{2}+4.4}/10^{\frac{3n}{2}+4.4} = 10^{3/2} \approx 31.6$

These computations indicate that each 1 unit increase in magnitude represents 31.6 times more energy released. (This result may seem to contradict other sources which state that each 1 unit increase in magnitude represents an earthquake that is 10 times stronger. However, those computations use *amplitude* as a measure of strength. The computations above are based on the energy released by the earthquake).

EXAMPLE 6: Interest compounded annually.

Using the formula $A = P(1 + r)^n$ (Lesson 12, Example 5) where *A* is the future value of the investment, *P* is the principal, *r* is the fixed annual interest rate, and *n* is the number of years, how many years will it take an investment to double if the interest rate per annum is 2.5%?

Solution:

Doubling the principal *P*, we get *A* = 2*P*, *r* = 2.5% = 0.025,

$$A = P(1+r)^n$$

$$2P = P(1+0.025)^n$$

$2 = (1.025)^n$

$\log 2 = \log(1.025)^n$

$\log 2 = n\log(1.025)$

$n = \dfrac{\log 2}{\log 1.025} \approx 28.07$ years

Answer: It will take approximately 28 years for investment to double.

Example 7. (Population growth) The population of the Philippines can be modeled by the function $P(x) = 20{,}000{,}000e^{0.0251x}$, where x is the number of years since 1955 (e.g. x = 0 at 1955). Assuming that this model is accurate, in what year will the population reach 200 million?

Solution.

Given P(x) = 200,000,000,

$200{,}000{,}000 = 20{,}000{,}000e^{0.0251x}$

$10 = e^{0.0251x}$

$\ln 10 = \ln e^{0.0251x}$

$\ln 10 = 0.0251x(\ln e)$

$\ln 10 = 0.0251x$

$x = \ln 10/0.0251 \approx 91$ years

1955 + 91 = 2046

Answer. Around the year 2046, the Philippine population will reach 200 million.

Trivia: Based on this model, we will reach 100 million in the year 2019. But last July 2014, the Philippines officially welcomed its 100 millionth baby[8]. Hence mathematical models must always be reviewed and verified against new data.

[8] http://newsinfo.inquirer.net/623749/philippines-welcomes-100-millionth-baby

EXAMPLE 8. In a bacteria culture, an initial population of 5,000 bacteria grows to 12,000 after 90 minutes. Assume that the growth of bacteria follows an exponential model $f(t) = Ae^{kt}$ representing the number of bacteria after t minutes. **(a)** Find A and k, and **(b)** use the model to determine the number of bacteria after 3 hours.

Solution.

(a) It is given that f(0) = 5,000 and f(90) = 12,000.

Thus, $f(0) = Ae^{k(0)} = A = 5{,}000$.

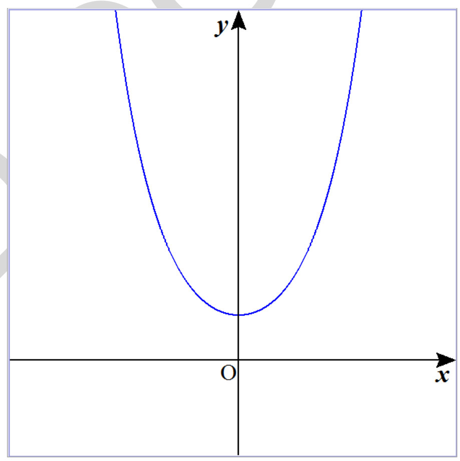
Also, $f(90) = 5{,}000e^{k(90)} = 12{,}000 \Rightarrow e^{90k} = \frac{12}{5}$.

Take the ln of both sides to obtain $\ln e^{90k} = \ln\frac{12}{5} \Rightarrow$ $90k = \ln\frac{12}{5} \Rightarrow k \approx 0.00973$

The exponential model is $f(t) = 5{,}000e^{0.00973t}$.

(b) 3 hours = 180 minutes; $f(180) = 5{,}000e^{0.00973(180)} \approx 28{,}813$ bacteria

Example 9. Chains or cables suspended between two points and acted upon by a gravitational force follow the shape of a catenary.[9] The equation $y = \frac{e^x+e^{-x}}{2}$ is an example of such a curve (see figure). Assuming this is the curve assumed by a suspended cable, how far apart are the cables when y = 4? Approximate your answer to two decimal places.



Solution. We have to solve the equation $4 = \frac{e^x+e^{-x}}{2}$ or $8 = e^x + e^{-x}$.

$$8 = e^x + e^{-x}$$

$$8 = e^x + \frac{1}{e^x}$$

$$8e^x = e^{2x} + 1$$

$$e^{2x} - 8e^x + 1 = 0$$

Let $u = e^x$. Then $u^2 = e^{2x}$, and we obtain $u^2 - 8u + 1 = 0$.

[9]Weisstein, Eric W. "Catenary." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/Catenary.html

The solutions to this quadratic equation are $u = 4 \pm \sqrt{15}$.

Since u = e^x then $4 + \sqrt{15} = e^x$ or $4 - \sqrt{15} = e^x$

$\ln(4 + \sqrt{15}) = x$ or $\ln(4 - \sqrt{15}) = x$

To determine the distance between the cables at y = 4, we find the difference between the two obtained x-values.

The distance is $\ln(4 + \sqrt{15}) - \ln(4 - \sqrt{15}) \approx 4.13$.

Solved Examples

1. Find the value/s of x in the following equations/inequalities.

a. $\log 15\text{x} = \log 30$

Solution.

$\log 15\text{x} = \log 30$

15x = 30

x = 2

b. $\log(3\text{x} - 2) = \log 2$

Solution.

$\log(3\text{x} - 2) = 2$

$10^2 = 3x - 2$

3x – 2 =100

$x = \frac{102}{3}$

c. $\log_{\text{x}} 121 = 2$

Solution.

$\log_{\text{x}} 121 = 2$

$x^2 = 121$

$x^2 - 121 = 0$

(x + 11)(x – 11) = 0

X = –11, 11

Since $\log_{-11} 121$ is not defined, hence, x = 4 is the only solution.

d. $\log_3(9x) - \log_3(x-8) = 4$

Solution.

$\log_3(9x) - \log_3(x-8) = 4$

$\log_3\left(\frac{9x}{x-8}\right) = 4$

$\frac{9x}{x-8} = 3^4$

$\frac{9x}{x-8} = 81$

9x = 81(x-8)

9x = 81x – 648

–72x = –648

x = 9

e. $(\log_5 x)^2 + 5\log_5 x + 6 = 0$

Solution.

Let $\log_5 x$ be equal to A

$(A)^2 + 5A + 6 = 0$

(A + 2)(A + 3) = 0

A = –2 or A = –3

$\log_5 x = -2$ or $\log_5 x = -3$

$x = 5^{-2} = \frac{1}{25}$ or $x = 5^{-3} = \frac{1}{125}$

f. $\log_8(3x-5) < 2$

Solution.

Ensuring that the logarithms are defined, this means 3x – 5 > 0 or x > 5/3.

$\log_8(3x-5) < 2$

$\log_8(3x-5) < 2\log_8 8$

$\log_8(3x-5) < \log_8 8^2$

$\log_8(3x-5) < \log_8 64$

$3x - 5 < 64$

$3x < 69$

$x < 23$

$\frac{5}{3} < x < 23$ (note that $x > 5/3$ from the first condition)

Therefore, the solution is (5/3, 23).

g. $\log_4(x + 1) < \log_4 2x$

Solution.

Ensuring that the logarithms are defined, this means $x + 1 > 0$ and $2x > 0$, which implies, $x > -1$ and $x > 0$, or just simply $x > -1$.

$\log_4(x + 1) < \log_4 2x$

$x + 1 < 2x$

$x - 2x < -1$

$-x < -1$

$x > 1$

Therefore, the solution is $(1, +\infty)$.

h. $-5 < \log x < 5$

Solution.

Ensuring the logarithms are defined, this means $x > 0$.

$-5 < \log x < 5$

$\log 10^{-5} < \log x < \log 10^5$

$\log x > \log 10^{-5}$ and $\log x < \log 10^5$

$x > 10^{-5}$ and $x < 10^5$

$\frac{1}{100{,}000} < x < 100{,}000$

Hence, the solution is $\left(\frac{1}{100{,}000}, 100{,}000\right)$.

3. When organisms die, the amount of carbon-14 in its system starts to decrease. The carbon-14 is about 5,600 years. If a piece of human bone was found to contain only 1/3 of the carbon-14 it originally had, how long ago did the human die?

Solution. A model for this situation is $y = y_0(1/2)^{t/5600}$, where y is the amount of carbon-14 in the organism after t years, and y0 is the initial amount of carbon-14.

Since $y = \frac{1}{3}y_0$ then $\frac{1}{3}y_0 = y_0\left(\frac{1}{2}\right)^{t/5600} \Longrightarrow \frac{1}{3} = \left(\frac{1}{2}\right)^{t/5600}$

Taking the ln of both sides, $\ln\frac{1}{3} = \ln\left(\frac{1}{2}\right)^{t/5600} = \frac{t}{5600}\ln\frac{1}{2}$.

$$\frac{\left(\ln\frac{1}{3}\right)}{\left(\ln\frac{1}{2}.\right)} = \frac{t}{5600} \Longrightarrow t = 5600\frac{\left(\ln\frac{1}{3}\right)}{\left(\ln\frac{1}{2}.\right)} \approx 8{,}876$$

The human died approximately 8,876 years ago.

4. A culture starts at 2,000 bacteria, and doubles every 80 minutes. How long will it take the number of bacteria to reach 10,000?

Solution. A model for this situation is $y = 2000(2)^{t/80}$, where y is the number of bacteria at time t.

If y = 10,000, then $10{,}000 = 2{,}000(2)^{t/80} \Rightarrow 5 = 2^{t/80}$.

Taking the ln of both sides, $\ln 5 = \ln 2^{t/80} = \frac{t}{80}\ln 2$.

Thus $\frac{\ln 5}{\ln 2} = \frac{t}{80} \Longrightarrow t = 80\frac{\ln 5}{\ln 2} \approx 185.7$

It will take approximately 185.7 or 186 minutes for the bacteria to reach 10,000.

5. Under certain circumstances, a rumor spreads according to the equation $p(t) = \frac{1}{1+15e^{-0.3t}}$, where p(t) is the proportion of the population who has heard of the rumor at time t days. How long will it take the rumor to reach 80% of the population?

Solution. Solve

$$0.8 = \frac{1}{1+15e^{-0.3t}}$$

$$0.8 + 12e^{-0.3t} = 1$$

$$12e^{-0.3t} = 0.2$$

$$e^{-0.3t} = 1/60$$

$$-0.3t = \ln(1/60) \Rightarrow t = \ln(1/60)/(-0.3) \approx 13.6$$

It will take approximately 14 days for the rumor to reach 80% of the population.

Lesson 21 Supplementary Exercises

Find the value/s of x in the following equations/inequalities.

1. $\log_3(x+4) = \log_3(2x-4)$

2. $\log_2(3x+5) \geq \log_2(x-9)$

3. $(\log_2 x)^2 - 4 = 0$

4. $\log x + \log(x-3) \leq \log 10$

5. An article from the New Yorker[10] states that the strongest earthquake that can be caused by the San Andreas fault has a magnitude of 8.2 on the Richter scale. It also states that this is only 6% as strong as the 2011 Japan earthquake, with a magnitude of 9.0. Verify this information.

6. Suppose that a fish population t days from now can be modeled by an exponential function $P(t) = Ae^{kt}$. Suppose that the fish population doubled in 90 days. By how much will the fish have multiplied from its initial number after 120 days?

7. Suppose that I_0 and I denote the light intensity before and after going through a material. Then according to Beer-Lambert's law[11], $x = -\frac{1}{k}\ln\left(\frac{I}{I_0}\right)$, where k is a constant that depends on the material. If k = 0.03 and I_0 = 15 lumens, find the line intensity at a depth of 30 feet.

8. The dapdap tree in the Philippines is prone to wasp infestation. Suppose that the percent P of defoliation is approximated by $P = \frac{300}{3+17e^{-0.0625x}}$, where x is the number of egg masses in thousands. If the percent of defoliation is 20%, approximately how many egg masses are there?

Lesson 22: Graphing Logarithmic Functions

Learning Outcome(s): At the end of the lesson, the learner is able to represent a logarithmic function through its table of values, graph, and equation, find the domain and range of a logarithmic function, and graph logarithmic functions

Lesson Outline:

1. Graph of $y = \log_b x$ for b > 1 and for 0 < b < 1
2. Domain, range, intercepts, zeroes, and asymptotes of logarithmic functions
3. Graphs of transformations of logarithmic functions

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[10] Schulz, K. (2015). *The really big one.* http://www.newyorker.com/magazine/2015/07/20/the-really-big-one

[11] Stewart, J., Redlin, L., & Watson, S. (2012). *Precalculus: Mathematics for calculus* (6th ed). Belmont, CA: Brooks/Cole, Cengage Learning.

In the following examples, the graph is obtained by first plotting a few points. Results will be generalized later on.

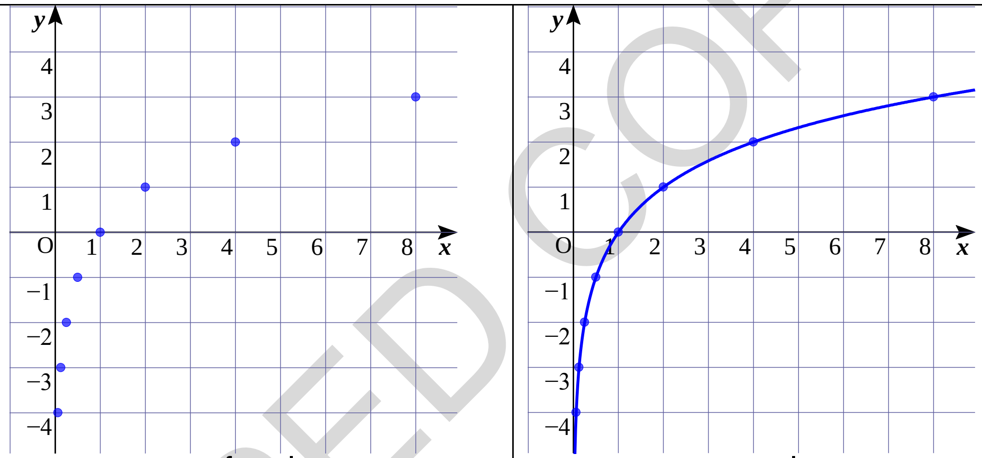
Example 1. Sketch the graph of $y = \log_2 x$.

Solution.

Step 1: Construct a table of values of ordered pairs for the given function. A table of values for $y = \log_2 x$ is as follows:

x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
y	–4	–3	–2	–1	0	1	2	3

Step 2: Plot the points found in the table, and connect them using a smooth curve.



Plotting of points for of $y = \log_2 x$

Graph of $y = \log_2 x$

It can be observed that the function is defined only for $x > 0$. The function is strictly increasing, and attains all real values. As x approaches 0 from the right, the function decreases without bound, i.e., the line $x = 0$ is a vertical asymptote.

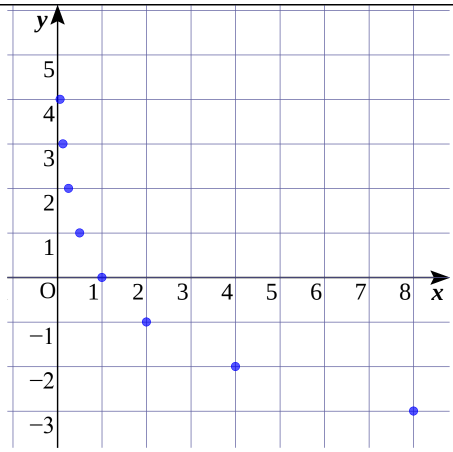
Example 2. Sketch the graph of $y = \log_{1/2} x$.

Solution.

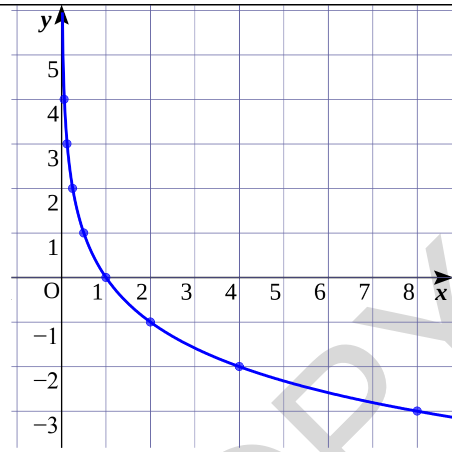
Step 1: Construct a table of values of ordered pairs for the given function. A table of values for $y = \log_2 x$ is as follows:

x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
y	4	3	2	1	0	–1	–2	–3

Step 2: Plot the points found in the table, and connect them using a smooth curve.



Plotting of points for of $y = \log_{1/2}x$

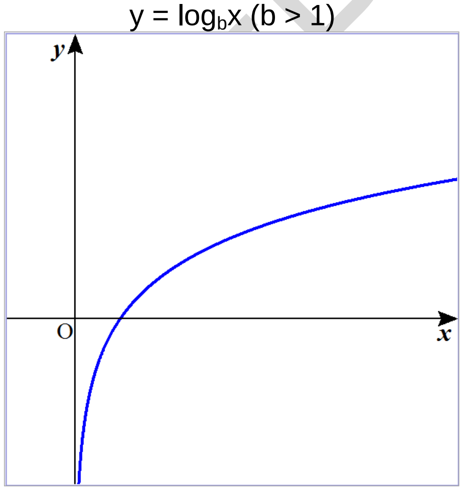


Graph of $y = \log_{1/2}x$

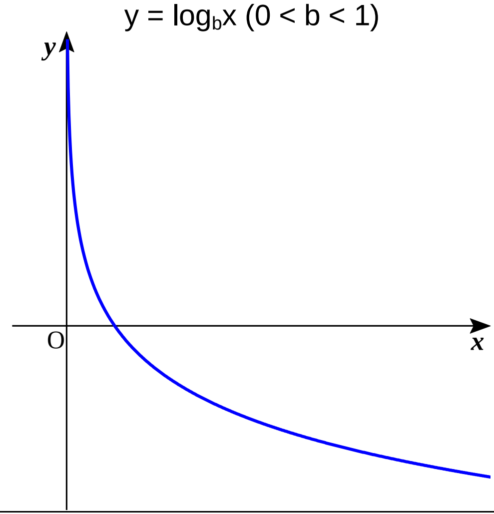
It can be observed that the function is defined only for $x > 0$. The function is strictly decreasing, and attains all real values. As x approaches 0 from the right, the function increases without bound, i.e., the line $x = 0$ is a vertical asymptote.

In general, the graphs of $y = \log_b x$, where $b > 0$ and $b \neq 1$ are shown below.

$y = \log_b x$ $(b > 1)$



$y = \log_b x$ $(0 < b < 1)$

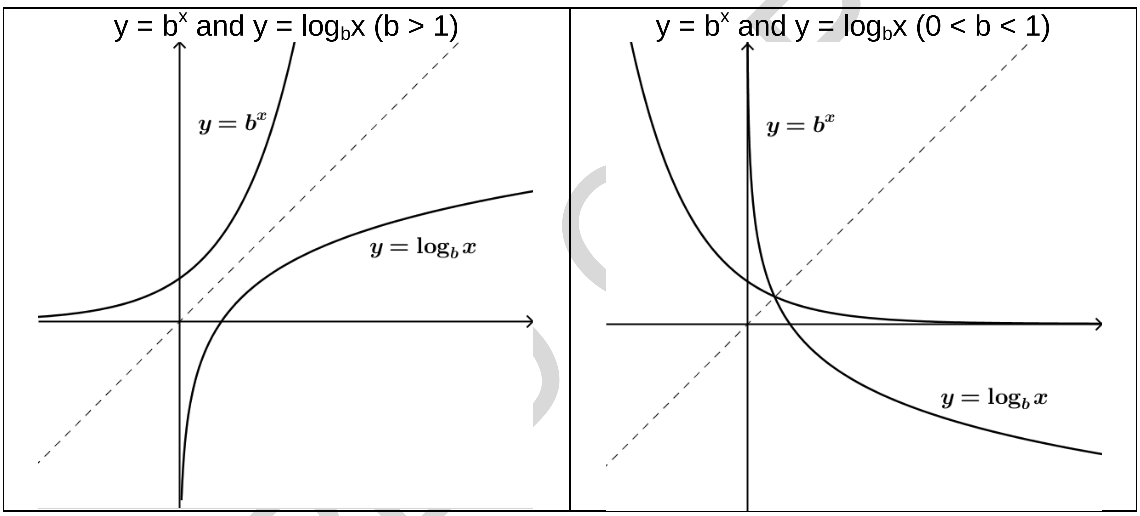


PROPERTIES OF LOGARITHMIC FUNCTIONS:

1. The domain is the set of all positive numbers, or $\{x \in \mathbb{R} \mid x > 0\}$.
2. The range is the set of all positive real numbers.
3. It is a one-to-one function. It satisfies the Horizontal Line Test.
4. The x-intercept is 1. There is no y-intercept.
5. The vertical asymptote is the line $x = 0$ (or the y-axis). There is no horizontal asymptote.

Relationship Between the Graphs of Logarithmic and Exponential Functions

Since logarithmic and exponential functions are inverses of each other, their graphs are reflections of each other about the line $y = x$, as shown below.



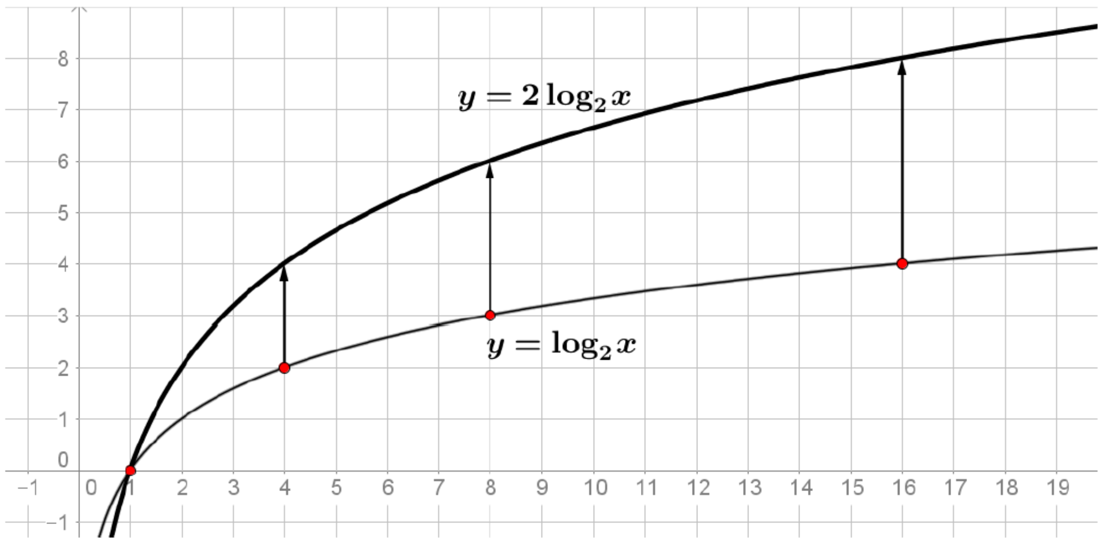
Example 3. Sketch the graphs of $y = 2\log_2 x$. Determine the domain, range, vertical asymptote, x-intercept, and zero.

Solution.

The graph of $y = 2\log_2 x$ can be obtained from the graph of $y = \log_2 x$ by multiplying each y-coordinate by 2, as the following table of signs shows.

x	1/16	1/8	¼	½	1	2	4	8
$\log_2 x$	–4	–3	–2	–1	0	1	2	3
$y = 2\log_2 x$	–8	–6	–4	–2	0	2	4	6

The graph is shown below.



Analysis:

a. Domain: {x | x ∈ ℝ, x > 0}

b. Range : {y | y ∈ ℝ }

c. Vertical Asymptote: x = 0

d. x-intercept: 1

e. Zero: 1

Example 4. Sketch the graph of $y = \log_3 x - 1$.

Solution.

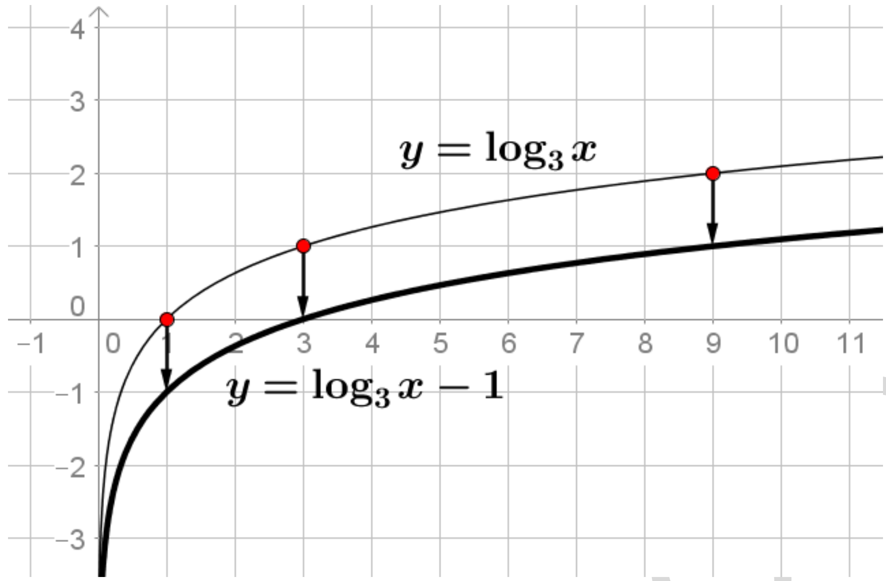
Sketch the graph of the basic function $y = \log_3 x$. Note that the base 3 > 1.

The "–1" means vertical shift downwards by 1 unit.

Some points on the graph of $y = \log_3 x$ are (1,0), (3,1), and (9,2).

Shift these points 1 unit down to obtain (1, –1), (3,0), and (9,1). **Plot these points.**

The graph is shown below.



Analysis:

a. Domain: {x | x ∈ ℝ, x > 0}

b. Range: {y | y ∈ ℝ }

c. Vertical Asymptote: x = 0

d. x-intercept: 3

The x-intercept can be obtained graphically. Likewise, we can solve for the x-intercept algebraically by setting y = 0:

$0 = \log_3 x - 1$

$\log_3 x = 1$

$x = 3^1 = 3$

e. Zero: 3

Example 5. Sketch the graph of $y = \log_{0.25}(x + 2)$.

Solution.

Sketch the graph of the basic function $y = \log_{0.25} x$.Note that the base $0 < 0.25 < 1$.

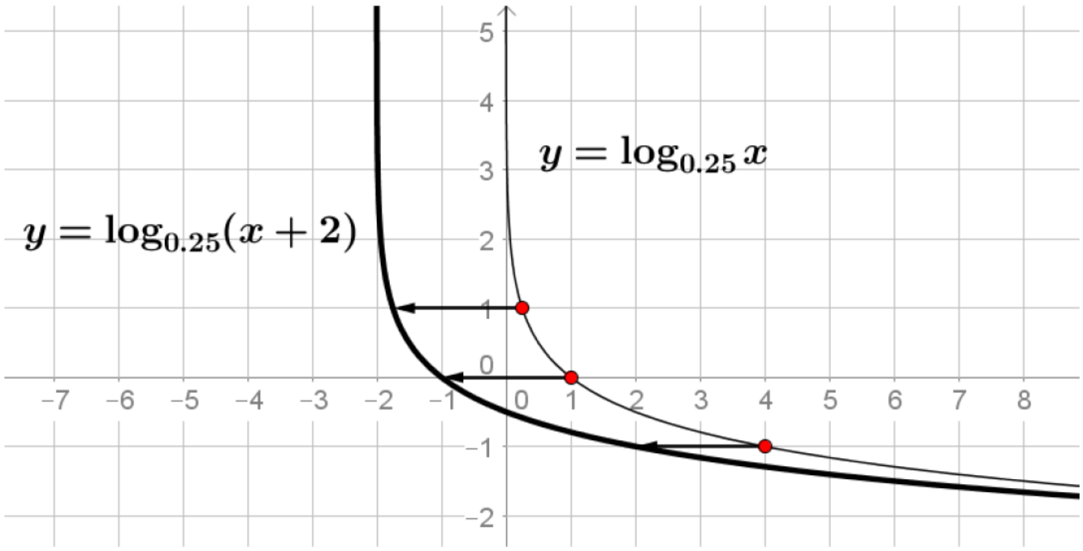
Rewrite the equation, obtaining $y = \log_{0.25}[x - (-2)]$.

The "–2" means a horizontal shift of 2 units to the left.

Some points on the graph of $y = \log_{0.25} x$ are (1,0), (4, –1), and (0.25,1).

Shift these points 2 units to the left to obtain (–1,0), (2, –1), and (–1.75,1). **Plot these points.**

Graph:



Analysis:

a. Domain: {x | x ∈ ℝ, x >–2}

(The expression x+2 should be greater than 0 for $\log_{0.25}(x+2)$ to be defined. Hence, x must be greater than –2.)

b. Range : {y | y ∈ ℝ }

c. Vertical Asymptote: x = –2

d. x-intercept: –1

e. Zero: –1

The examples above can be generalized to form the following guidelines for graphing transformations of logarithmic functions:

Graph of $f(x) = a \cdot \log_b(x - c) + d$

- The value of b (either b > 1 or 0 < b < 1) determines whether the graph is increasing or decreasing.
- The value of *a* determines the stretch or shrinking of the graph. Further, if *a* is negative, there is a reflection of the graph about the x-axis.
- Based on $y = a \cdot \log_b x$, the vertical shift is d units up (if d > 0) or d units down (if d < 0), and the horizontal shift is c units to the right (if c > 0) or c units to the left (if c < 0).

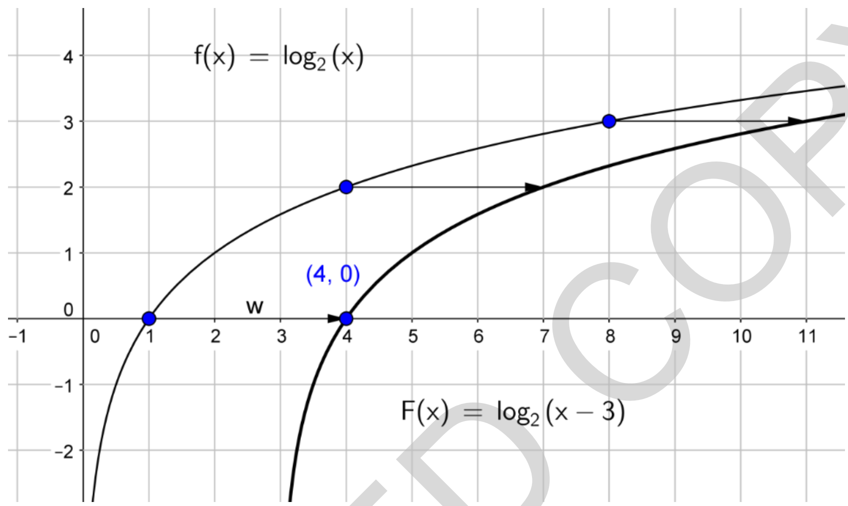
Solved Examples

Analyze each of the following functions by (a) using the transformations to describe how the graph is related to a logarithmic function y = $\log_b x$, (b) identifying the x-intercept, vertical asymptote, domain and range. (c) Sketch the graph of the function.

a.) F(x) = $\log_2(x - 3)$

Solution.

The graph of F(x) is shifted 3 units to the right from the graph of f(x) = $\log_2(x)$.



Domain: { x ∈ ℝ | x > 3 }

Range: all real numbers

Vertical Asymptote: x = 3

x-intercept: (4, 0)

b.) G(x) = $\log_{0.5}(x) - 3$

Solution.

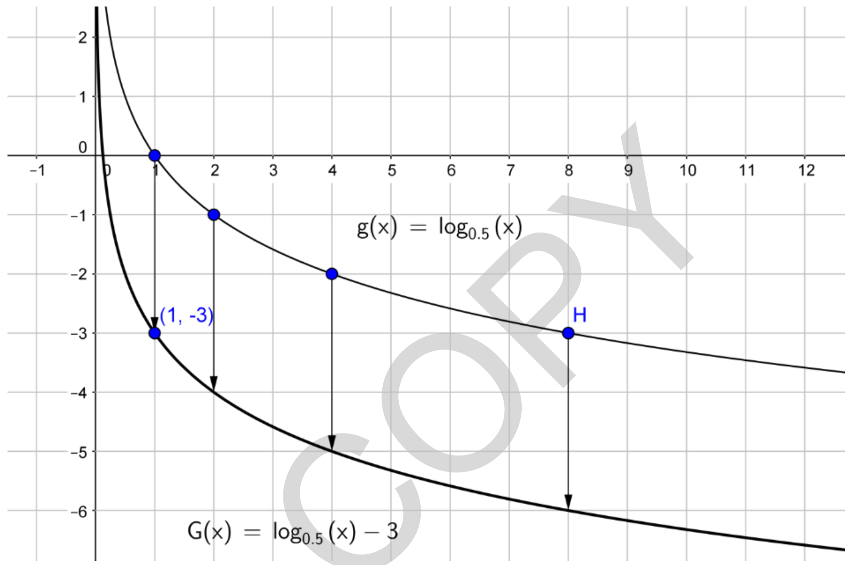
The graph of G(x) is a vertical shift of 3 units downwards from the graph of g(x) = $\log_{0.5}(x)$.

Domain: { x ∈ ℝ | x > 0 }

Range: all real numbers

Vertical Asymptote: x = 0

x-intercept: (0.125, 0)



c.) H(x) =$3\log_2 x$

Solution.

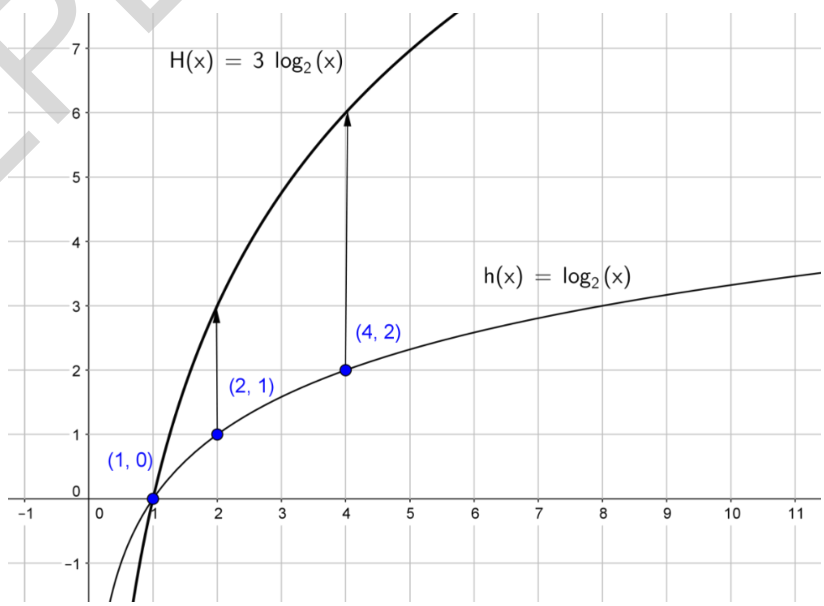
The graph of H(x) is a stretch by a factor of 3 from the graph of h(x) = $\log_2(x)$.

Domain: { x ∈ ℝ | x > 0 }

Range: all real numbers

Vertical Asymptote: x = 0

x-intercept: (1, 0)



The previous examples can be generalized to form the following guidelines for graphing transformations of logarithmic functions:

Graph of $f(x) = a \cdot \log_b(x - c) + d$

- The value of b (either $b > 1$ or $0 < b < 1$) determines whether the graph is increasing or decreasing.
- The value of *a* determines the stretch or shrinking of the graph. Further, if *a* is negative, there is a reflection of the graph about the x-axis.
- Based on $y = a \cdot \log_b x$, the vertical shift is d units up (if $d > 0$) or d units down (if $d < 0$), and the horizontal shift is c units to the right (if $c > 0$) or c units to the left (if $c < 0$)

Lesson 22 Supplementary Exercises

Analyze each of the following functions by (a) using the transformations to describe how the graph is related to a logarithmic function $y = \log_b x$, (b) identifying the x-intercept, vertical asymptote, domain and range. (c) Sketch the graph of the function.

a. $F(x) = \log_{0.25}(x - 3)$

b. $G(x) = \log_3(x + 2) - 2$

c. $H(x) = \log_2(x - 1) + 2$

Topic Test 1

1. Find the value of the following logarithmic expressions: **[5]**

   a. $\log_4(1/64)$ b. $\log_{1/2}64$

2. Express logx + 2logy – 3loga as a single logarithm. **[5]**

3. Solve for x. **[10]**

   a. $2\log_7 x = \log_7 81$ b. $\ln(x+2) + \ln(x-1) = \ln(9x-17)$

4. Solve the inequality $\log_4 x + 8 \geq 11$ **[10]**

5. If a certain sound wave has an intensity of 10^{-5} W/m^2, find its corresponding decibel value.
   [10]

6. Graph the following functions. Label all intercepts and asymptotes. Indicate the domain and range. **[10]**

   a. $f(x) = \log_2(x) + 4$ b. $g(x) = 2\log_3(x-1)$

Topic Test 2

1. Express $\ln xy^2$ as a sum, difference or multiple of logarithms. **[5]**

2. Express $\log_5 6$ as a quotient of logarithms to the base 2. **[5]**

3. Solve for x in the following equations. **[10]**

a. $4\log_5 x = \log_5 16$ b. $2\log_{49}(2x+1) - 1 = 0$

4. Solve the inequality $\log_{1/2}(2x+3) > \log_{1/2}(x-2)$ **[10]**

5. Felix deposited ₱50,000 in an investment that earns 5% interest annually. How many years will his investment be equal to ₱175,000? **[10]**

6. Graph the following functions. Label all intercepts and asymptotes. Indicate the domain and range. **[10]**

a. $f(x) = \log_3(x) - 1$ b. $g(x) = 4\log_2(x+3)$

Lesson 23: Illustrating Simple and Compound Interest

Learning Outcome(s): At the end of the lesson, the learner is able to illustrate simple and compound interest and distinguish between simple and compound interest.

Lesson Outline:

1. Simple Interest
2. Compound Interest

Definitions of Terms:

Lender or creditor–person (or institution) who invests the money or makes the funds available

Borrower or debtor – person (or institution) who owes the money or avails of the funds from the lender

Origin or **loan date** – date on which money is received by the borrower

Repayment date or **maturity date** –date on which the money borrowed or loan is to be completely repaid

Time or term (t)– amount of time in years the money is borrowed or invested; length of time between the origin and maturity dates

Principal (P)– amount of money borrowed or invested on the origin date

Rate(r)– annual rate, usually in percent, charged by the lender, or rate of increase of the investment

Interest (I)– amount paid or earned for the use of money

Simple Interest (I_s) – interest that is computed on the principal and then added to it

Compound Interest (I_c)–interest is computed on the principal and also on the accumulated past interests

Maturity value or future value (F) –amount after *t* years that the lender receives from the borrower on the maturity date

Illustration of Simple and Compound Interest

Example 1. "Suppose you won 10,000 pesos and you plan to invest it for 5 years. A cooperative group offers 2% simple interest rate per year. A bank offers 2% compounded annually. Which will you choose and why?"

Solution.

Investment 1: Simple Interest

Time (t)	Principal (P)	Interest Rate (r)	**Simple Interest**		Amount after *t* years (Maturity Value)
			Solution	Answer	
1	10,000	2%	(10000)(0.02)(1)	200	10 000 + 200 = 10 200.00
2		2%	(10000)(0.02)(2)	400	10 000 + 400 = 10 400.00
3		2%	(10000)(0.02)(3)	600	10 000 + 600 = 10 600.00
4		2%	(10000)(0.02)(4)	800	10 000 + 800 =10 800.00
5		2%	(10000)(0.02)(5)	1 000	10 000 + 1 000 = 11 000.00

Investment 2: Compound Interest (Annual)

Time (t)	Amount at the start of year *t*	Rate (r)	Compound Interest		Amount at the end of year *t* (Maturity Value)
			Solution	Answer	
1	10 000	2%	(10000)(0.02)(1)	200	10 000 + 200 = 10 200.00
2	10 200	2%	(10200)(0.02)(1)	204	10 200 + 204 = 10 404.00
3	10 404	2%	(10404)(0.02)(1)	208.08	10 404 + 208.08 =10 612.08
4	10 612.08	2%	(10612.08)(0.02)(1)	212.24	10612.08 + 212.24 =10 824.32
5	10 824.32	2%	(10824.32)(0.02)(1)	216.49	10824.32 + 216.49 = 11 040.81

Simple interest remains constant throughout the investment term. In compound interest, the interest from the previous year also earns interest. Thus, the interest grows every year.

Lesson 24: Simple Interest

Learning Outcome(s): At the end of the lesson, the learner is able to compute interest, maturity value, and present value in simple interest environment, and solve problems involving simple interest

Lesson Outline:

1. Compute simple interest
2. Compute maturity value
3. Compute unknown principal, rate, or time

Annual Simple Interest

$$I_s = Prt$$

where
I_s = simple interest
P = principal, or the amount invested or borrowed
r = simple interest rate
t = term or time in years

Example 1: A bank offers 0.25% annual simple interest rate for a particular deposit. How much interest will be earned if 1 million pesos is deposited in this savings account for 1 year?

Given: P = 1,000,000 I = 0.25% = 0.0025 t = 1 year

Find: I_s

Solution: $I_s = Prt$
$I_s = (1{,}000{,}000)(0.0025)(1)$
$I_s = 2{,}500$

Answer: The interest earned is P 2,500.

Example 2: How much interest is charged when P50,000 is borrowed for 9 months at an annual interest rate of 10%?

Given: P = 50,000 r = 10% = 0.10 $t = \frac{9}{12}$ year = 0.75 year

Find: I_s

Note: When the term is expressed in months (M), it should be converted in years by $t = \frac{M}{12}$.

Solution: $I_s = Prt$

$I_s = (50{,}000)(0.10)(\frac{9}{12})$

$I_s = (50{,}000)(0.10)(0.75)$

$I_s = 3{,}750$

Answer: The simple interest charged is P 3,750.

Example 3: Complete the table below by finding the unknown.

Principal (P)	Rate (r)	Time (t)	Interest
(a)	2.5%	4	1,500
36,000	(b)	1.5	4,860
250,000	0.5%	(c)	275
500,000	12.5%	10	(d)

Solution:

(a) The unknown principal can be obtained by

$P = \frac{I_s}{rt}$

$P = \frac{1{,}500}{(0.025)(4)}$

$P = 15{,}000$

(b) The unknown rate can be computed by

$r = \frac{I_s}{Pt}$

$r = \frac{4{,}860}{(36{,}000)(1.5)}$

$r = 0.09 = 9\%$

(c) The unknown time can be calculated by

$t = \frac{I_s}{Pr}$

$t = \frac{275}{(250{,}000)(0.005)}$

$t = 0.22$ years

(d) The unknown simple interest is given by

$I_s = Prt$

$I_s = (500{,}000)(0.125)(10)$

$I_s = 625{,}000$

Example 4: When invested at an annual interest rate of 7%, the amount earned P11,200 of simple interest in two years. How much money was originally invested?

Given: $r = 7\% = 0.07$ $t = 2$ years $I_S = 11,200$

Find: Amount invested or principal P

Solution:

$$P = \frac{I_S}{rt}$$

$$P = \frac{11,200}{(0.07)(2)}$$

$$P = 80,000$$

Answer: The amount invested is P 80,000.

Example 5: If an entrepreneur applies for a loan amounting to P500,000 in a bank, the simple interest of which is P 157,500 for 3 years, what interest rate is being charged?

Given: $P = 500,000$ $I_s = 157,500$ $t = 3$ years

Find: r

Solution:

$$r = \frac{I_S}{Pt}$$

$$r = \frac{157,500}{(500,000)(3)}$$

$$r = 0.105 = 10.5\%$$

Answer. The bank charged an annual simple interest rate of 10.5%.

Example 6: How long will a principal earn an interest equal to half of it at 5% simple interest?

Given: P $r = 5\% = 0.05$ $I_s = \frac{1}{2}P = 0.5P$

Find: t

Solution:

$$t = \frac{I_S}{Pr}$$

$$t = \frac{0.5P}{(P)(0.05)}$$

$$t = 10 \text{ years}$$

Answer: It will take 10 years for a principal to earn half of its value at 5% simple annual interest rate.

Maturity (Future) Value
$$\mathbf{F = P + I_s}$$
where F = maturity (future) value
P = principal
I_s = simple interest

Substituting I_s by Prt gives $F = P + Prt, = P(1 + rt)$

Maturity (Future) Value
$$\mathbf{F = P(1 + rt)}$$
where F = maturity (future) value
P = principal
r = interest rate
t = term/ time in years

Example 7: Find the maturity value if 1 million pesos is deposited in a bank at an annual simple interest rate of 0.25% after (a) 1 year (b) 5 years?

Given: P = 1,000,000 r = 0.25% = 0.0025
Find: (a) maturity or future value F after 1 year
(b) maturity or future value F after 5 years

Solution:

(a) When t = 1, the simple interest is given by

Method 1:
$I_s = Prt$
$I_s = (1,000,000)(0.0025)(1)$
$I_s = 2,500$

The maturity or future value is given by
$F = P + I_s$
$F = 1,000,000 + 2,500$
$F = 1,002,500$

Method 2: To directly solve the future value F,
$F = P(1 + rt)$
$F = (1{,}000{,}000)(1 + 0.0025(1))$
$F = 1{,}002{,}500$
Answer: The future or maturity value after 1 year is P1,002,500.

(b) When t = 5,

Method 1:
$I_s = Prt$
$I_s = (1{,}000{,}000)(0.0025)(5)$
$I_s = 12{,}500$

$F = P + I_s$
$F = 1{,}000{,}000 + 12{,}500$
$F = 1{,}012{,}500$

Method 2:
$F = P(1 + rt)$
$F = (1{,}000{,}000)(1 + 0.0025(5))$
$F = 1{,}012{,}500$

Answer: The future or maturity value after 5 years is P1,012,500.

Solved Examples:

1. What are the amounts of interest and maturity value of a loan for P25,000 at 12% simple interest for 5 years?
   Given: P = 25,000, r = 0.12, t = 5 years
   Find: (i) I_s
   (ii) F
   Solution:
   (i) : $I_s = Prt$
   $I_s = (25{,}000)(0.12)(5)$
   Answer: $I_s = 15{,}000$

   (ii) $F = P + I_s$
   $F = 25{,}000 + 15{,}000$

Answer: F = 40,000

2. How much money will you have after 4 years and 3 months if you deposited P 10,000 in a bank that pays 0.5% simple interest?

   Given: P = 10,000, r = 0.005, t = 4.25 years
   Find: F
   Solution: $F = P(1 + rt)$
   $F = (10{,}000)(1 + 0.005(4.25))$

Answer: F = 10,212.25

3. At what simple interest rate per annum will P 1 become P 2 in 2 years?
   Given: P = 1, F = 2, t = 1
   Find: r
   Solution: I_s = F – P = 2 -1 = 1

$$r = \frac{I_s}{Pt}$$

$$r = \frac{1}{(1)(2)}$$

$$r = 0.5$$

   Answer: r = 50%

4. How long will 1 million pesos earn a simple interest of 100,000 at 1% per annum?
   Given: P = 1,000,000 I_s = 100,000 r = 0.01

   Find: t

   Solution: $t = \frac{I_s}{Pr}$

$$t = \frac{100{,}000}{(1{,}000{,}000)(0.01)}$$

   Answer: t = 10 years

5. How much should you invest at the simple interest is 7.5% in order to have P300,000 in 2 years,?
   Given: r = 0.075 F = 300,000 t = 2
   Find: P
   Solution: $P = \frac{F}{1+rt}$

$$P = \frac{300{,}000}{1+(0.075)(2)}$$

   Answer: P = 260,869.57

Lesson 24 Supplementary Exercises

A. Find the unknown principal P, rate r, time t, and interest I by completing the table.

Principal (P)	Rate (r)	Time (t)	Interest (I)
2,000	5%	3	(1)
(2)	12%	5	20,000
88,000	(3)	4	8,000
500,000	9.5%	(4)	285,000
1,000,000	0.5%	12	(5)

B. Complete the table by finding the unknown.

Present Value (P)	Rate (r)	Time (t)	Interest (I)	Maturity Value (F)
8,000	8%	6 years	(6)	(7)
5,000	(8)	1 month	(9)	6000
(10)	12%	5 years and 3 months		400,000

C. Solve the following problems.

11. Find the simple interest on a loan of P65,000 if the loan is given at a rate of 20% and is due in 3 years.
12. Amparo invested a certain amount at 10% simple interest per year. After 2 years, the interest she received amounted to P3,000. How much did she invest?
13. Miko borrowed P25,000 at 10% annual simple interest rate. How much should he pay after 3 years and 6 months?
14. How long will an amount of money triple at a simple interest rate of 1% per annum?
15. How long will an amount of P50,000 gain a simple interest of P10,000 at 4% per annum?
16. At what simple interest rate will an amount of money gain 50% of the principal in 4 years?
17. At what simple interest rate per annum will P20,000 accumulate to P25,000 in 3 years?
18. If you deposit P5,500 in a bank at an annual simple interest rate of 0.5% , how much money will you have after 12 years?
19. How much money should you deposit in a bank so that it will accumulate to P100,000 at 1% simple annual interest for 10 years?
20. How much should you invest at 6% annual interest rate to obtain a simple interest of P72,000 in 3 years?

Lesson 25: Compound Interest

Learning Outcome(s): At the end of the lesson, the learner is able to compute interest, maturity value, and present value in compound interest environment, and solve problems involving compound interest

Lesson Outline:

1. Maturity value
2. Present Value

The following table shows the amount at the end of each year if principal P is invested at an annual interest rate r compounded annually. Computations for the particular example P = P100,000 and r = 5% are also included.

Year (t)	**Principal = *P* Int. rate =*r*, compounded annually**	**Principal = P 100,000 Int. rate =5%, compounded annually**
	Amount at the end of the year	**Amount at the end of the year**
1	$P \times (1 + r) = \boldsymbol{P(1 + r)}$	100,000 × 1.05 = **105,000**
2	$\boldsymbol{P(1 + r)} \times (1 + r) = \boldsymbol{P(1 + r)^2}$	**105,000**×1.05 = **110,250**
3	$\boldsymbol{P(1 + r)^2} \times (1 + r) = \boldsymbol{P(1 + r)^3}$	**110,250**× 1.05 = **121,550.63**
4	$\boldsymbol{P(1 + r)^3} \times (1 + r) = = \boldsymbol{P(1 + r)^4}$	**121,550.63**× 1.05 = **127,628.16**

Maturity (Future) Value and Compound Interest

$$F = P(1+r)^t$$

where
P = principal or present value
F = maturity (future) value at the end of the term
r = interest rate
t = term/ time in years

The compound interest I_c is given by

$$I_c = F - P$$

Example 1. Find the maturity value and the compound interest if P10,000 is compounded annually at an interest rate of 2% in 5 years.

Given: P = 10,000 r = 2% = 0.02 t = 5 years

Find: (a) maturity value F

(b) compound interest I_c

Solution:

(a) $F = P(1+r)^t$

$F = (10{,}000)(1 + 0.02)^5$

$F = 11{,}040.\ 081$

(b) $I_c = F - P$

$I_c = 11{,}040.81 - 10{,}000$

$I_c = 1{,}040.81$

Answer: The future value F is P11,040. 81 and the compound interest is P1,040.81.

Example 2. Find the maturity value and interest if P 50,000 is invested at 5% compounded annually for 8 years.

Given: P = 50,000 r = 5% = 0.05 t = 8 years

Find: (a) maturity value F

(b) compound interest I_c

Solution:

(a) $F = P(1+r)^t$

$F = (50{,}000)(1 + 0.05)^8$

$F = 73{,}872.77$

(b) $I_c = F - P$

$I_c = 73{,}872.77 - 50{,}000$

$I_c = 23{,}872.77$

Answer: The maturity value F is P73,872.77 and the compound interest is P23,872.77.

Example 3. Suppose your father deposited in your bank account P10,000 at an annual interest rate of 0.5% compounded yearly when you graduate from kindergarten and did not get the amount until you finish Grade 12. How much will you have in your bank account after 12 years?

Given: P = 10,000 r = 0.5% = 0.005 t = 12 years

Find: F

Solution: The future value F is calculated by

$F = P(1+r)^t$

$F = (10{,}000)(1 + 0.005)^{12}$

$F = 10{,}616.78$

Answer: The amount will become P10,616.77 after 12 years.

Present Value P at Compound Interest:

The present value or principal of the maturity value F due in t years any rate r can be obtained from the maturity value formula $F = P(1+r)^t$.

Solving for the present value P,

$P(1+r)^t = F$

$\frac{P(1+r)^t}{(1+r)^t} = \frac{F}{(1+r)^t}$

$P = \frac{F}{(1+r)^t}$ or equivalently, $P = F(1+r)^{-t}$

Present Value P at Compound Interest

$$\boldsymbol{P = \frac{F}{(1+r)^t} = F(1+r)^{-t}}$$

where

P = principal or present value
F = maturity (future) value at the end of the term
r = interest rate
t = term/ time in years

Example 4. What is the present value of P50,000 due in 7 years if money is worth 10% compounded annually?

Given: F = 50,000 r = 10% = 0.1 t = 7 years

Find: P

Solution: The present value P can be obtained by

$P = \frac{F}{(1+r)^t}$

$P = \frac{50{,}000}{(1+0.1)^7}$

P = 25,657.91.

Answer: The present value is P25,657.91.

Example 5. How much money should a student place in a time deposit in a bank that pays 1.1% compounded annually so that he will have P200,000 after 6 years?

Given: F = 200,000   r = 1.1% = 0.011   t = 6 years

Find: P

Solution: The present value P can be obtained by

$$P = \frac{F}{(1+r)^t}$$

$$P = \frac{200{,}000}{(1+0.011)^6}$$

P = 187,293.65

Answer: The student should deposit P187,293.65 in the bank.

Solved Examples

1. Mr. Ocampo invested P150,000 at 10% compounded annually. He plans to get this amount after 6 years for his son's college education. How much will he get?

   Given: P = P150,000   r = 10% = 0.1   t = 6 years
   Find: F

   Solution:

   $$F = P(1+r)^t$$

   $$F = (150{,}000)(1 + 0.1)^6$$

   Answer: He will get P265,734.15.

2. What is the interest of P25,000 if invested at 4.5% compounded annually in 3 years and 2 months?
   Given: P = P25,000   r = 4.5% = 0.045   t = $3\frac{2}{12}$ years
   Find: I_c
   Solution:

   $$F = P(1+r)^t$$
   $$F = (25{,}000)(1 + 0.045)^{3.1667}$$
   : $$F = 28{,}739.22$$

   $$I_c = F - P$$
   $$I_c = 28{,}739.22 - 25{,}000$$

   Answer: The interest is P3,739.22.

3. Mr. Bautista aims to have his investment grow to P500,000 in 4 years. How much should he invest in an account that pays 5% compounded annually?

Given: F = 500,000 r = 0.05 t = 4 years
Find: P
Solution: $P = \frac{F}{(1+r)^t}$

$P = \frac{500,000}{(1+0.05)^4}$

P = 411,351.24

Answer: He should invest P411,351.24.

4. Mrs. Versoza wants to compare simple and compound interests on a P350,000 investment for 3 and 3 months years.
   a. Find the interest if funds earn 6.5% simple interest for 1 year.
   b. Find the interest if funds earn 6.5% interest compounded annually.
   c. Find the difference between the two interests.

Given: P = 350,000 r = 0.065 t = 3.25 years

Find: a. I_s

b. I_c

c. Difference between I_s and I_c

Solution:

a. $I_s = Prt$
$I_s = (350,000)(0.065)(3.25)$

Answer: The interest is P73,937.50.

b. $F = P(1+r)^t$
$F = (350,000)(1 + 0.065)^{3.25}$

F = 429,491.20

$I_c = F - P$

$I_c = 429,491.20 – 350,000$

Answer: The interest is P79,491.20.

c. Difference = $I_c – I_s$
Difference = 79,491.20 –73,937.50 = 5,553.70

Answer: The difference between the two interests is P5,553.70.

Lesson 25 Supplementary Exercises

A. Find the unknown principal P, rate r, time t, and compound interest I_c by completing the table.

Principal (P)	Rate (r)	Time (t)	Compound Interest (I_c)	Maturity Value (F)
6,000	8%	12	(1)	(2)
12,000	5.5%	6 years and 9 months	(3)	(4)
60,000	9.75%	10 months	(5)	(6)
(7)	1%	6	(8)	25,000
(9)	7.5%	4 years and 6 months	(10)	400,000

B. Solve the following problems on compound interest.

11. Peter borrowed P100,000 at 8% compounded annually? How much will he be paying after 2 years?

12. A time deposit account in a bank yields 5.5% compound interest annually. Jennifer invested P450,000 for 4 years in this savings account. How much interest will she gain?

13. In order to have P250,000 in 5 years, how much should you invest if the compound interest is 12%?

14. How much money must be invested to obtain an amount of P150,000 in 2 years if money earns at 10.5% compounded annually?

15. What amount must be deposited by a student in a bank that pays 2 % compounded annually so that after 12 years he will have P100,000?

For numbers 16 to 18: Nora is thinking of investing an amount of P30,000 for 2 ½ years. Find the future value based on the following investments:

16. Simple interest of 8.5%
17. 8.5% compounded annually
18. Which investment is better? Justify your answer.

For numbers 19 & 20: Kaye aims to accumulate an amount of P180,000 in 5 years and 6 months. Find the present value based on the following investments and tell which investment requires a smaller principal.

19. Simple interest of 8.5%
20. 8.5% compounded annually

Lesson 26: Compounding More than Once a Year

Learning Outcome(s): At the end of the lesson, the learner is able to compute maturity value, interest, and present value, and solve problems involving compound interest when compound interest is computed more than once a year

Lesson Outine:

1. Compounding more than once a year
2. Maturity value, interest, and present value when compound interest is computed more than once a year

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Sometimes, interest may be compounded more than once a year. Consider the following example.

Example 1. Given a principal of PhP 10,000, which of the following options will yield greater interest after 5 years:

OPTION A: Earn an annual interest rate of 2% at the end of the year, or

OPTION B: Earn an annual interest rate of 2% in two portions—1% after 6 months, and 1% after another 6 months?

Solution.

OPTION A: Interest is compounded *annually*

Time (t) in years	Principal = PhP 10,000 Annual Int. rate = 2%, compounded annually
	Amount at the end of the year
1	$10{,}000 \times 1.02 = 10{,}200$
2	$10{,}200 \times 1.02 = 10{,}404$
3	$10{,}404 \times 1.02 = 10{,}612.08$
4	$10{,}612.08 \times 1.02 = 10{,}824.32$
5	$10{,}824.32 \times 1.02 = 11{,}040.81$

OPTION B: Interest is compounded *semi-annually,* or every 6 months.

Under this option, the interest rate every six months is 1% (2% divided by 2).

Time (t) in years	Principal = PhP 10,000 Annual Int. rate = 2%, compounded semi-annually Amount at the end of the year
½	10,000 × 1.01 = **10,100**
1	**10,100**× 1.01 = **10,201**
1½	**10,201**× 1.01 = **10,303.01**
2	**10,303.01** × 1.01 = **10,406.04**
2½	**10,406.04** × 1.01 = **10,510.10**
3	**10,510.10** × 1.01 = **10,615.20**
3½	**10,615.20** × 1.01 = **10,721.35**
4	**10,721.35** × 1.01 = **10,828.56**
4½	**10,828.56**× 1.01 = **10,936.85**
5	**10,936.85** × 1.01 = **11,046.22**

Answer: Option B will give the higher interest after 5 years.If all else is equal, **a more frequent compounding will result in a higher interest,** which is why Option B gives a higher interest than Option A.

The investment scheme in Option B introduces new concepts because interest is compounded twice a year, the **conversion period** is 6 months, and the **frequency of conversion** is 2. As the investment runs for 5 years, the **total number of conversion periods** is 10. The **nominal rate** is 2% and the **rate of interest for each conversion period** is 1%. These terms are defined generally below.

Definition of Terms:

Frequency of conversion (m) – number of conversion periods in one year

Conversion or interest period– time between successive conversions of interest

Total number of conversion periods n

$$n = mt = (\text{frequency of conversion}) \times (\text{time in years})$$

Nominal rate ($i^{(m)}$) – annual rate of interest

Rate (j) of interest for each conversion period

$$j = \frac{i^{(m)}}{m} = \frac{\text{annual rate of interest}}{\text{frequency of conversion}}$$

Note on rate notation: r, $i^{(m)}$, j

In earlier lessons, r was used to denote the interest rate. Now that an interest rate can refer to two rates (either nominal or rate per conversion period), the symbols $i^{(m)}$ and j will be used instead.

Examples of nominal rates and the corresponding frequencies of conversion and interest rate for each period:

$i^{(m)}$ = Nominal Rate (Annual Interest Rate)	m = Frequency of Conversions	j = Interest Rate per conversion period	One conversion period
2% compounded annually; $i^{(1)}$ =0.02	1	$\frac{0.02}{1}$ = 0.02 = 2%	1 year
2% compounded semi-annually; $i^{(2)}$ = 0.02	2	$\frac{0.02}{2}$ = 0.01 = 1%	6 months
2% compounded quarterly; $i^{(3)}$ = 0.02	4	$\frac{0.02}{4}$ = 0.005 = 0.5%	3 months
2% compounded monthly; $i^{(12)}$= 0.02	12	$\frac{0.02}{12}$ = $0.001\bar{6}$ = $0.1\bar{6}$%	1 month
2% compounded daily; $i^{(365)}$= 0.02	365	$\frac{0.02}{365}$	1 day

From Lesson 25, the formula for the maturity value F when principal P is invested at an annual interest rate *j* compounded annually is **$F= P(1+j)^t$**.

Because the rate for each conversion period is $j = \frac{i^{(m)}}{m}$, then in *t* years, interest is compounded *mt* times. The following formula is obtained.

Maturity Value, Compounding m times a year

$$F = P\left(1 + \frac{i^{(m)}}{m}\right)^{mt}$$

where F = maturity (future) value
P = principal
$i^{(m)}$ = nominal rate of interest (annual rate)
m = frequency of conversion
t = term/ time in years

Example 2. Find the maturity value and interest if P10,000 is deposited in a bank at 2% compounded quarterly for 5 years.

Given: P = 10,000 $i^{(4)} = 0.02$ t = 5 years m = 4

Find: a. F

b. P

Solution.

Compute for the interest rate in a conversion period by

$$j = \frac{i^{(4)}}{m} = \frac{0.02}{4} = 0.005.$$

Compute for the total number of conversion periods given by

n = mt = (4)(5) = 20 conversion periods.

Compute for the maturity value using

$F = P(1+j)^n$

$= (10,000)(1 + 0.005)^{20}$

F= 11,048.96.

The compound interest is given by

$I_c = F - P = 11,048.96 - 10,000 = P1,048.96.$

Example 3. Find the maturity value and interest if P10,000 is deposited in a bank at 2% compounded monthly for 5 years.

Given: P = 10,000 $i^{(12)} = 0.02$ t = 5 years m= 12

Find: a. F

b. P

Solution.

Compute for the interest rate in a conversion period by

$$j = \frac{i^{(12)}}{m} = \frac{0.02}{12}$$

Compute for the total number of conversion periods given by

n = mt = (12)(5) = 60 conversion periods.

Compute for the maturity value using

$F = P(1+j)^n$

$= (10\ 000)\left(1 + \frac{.02}{12}\right)^{60}$

F = P11,050.79

.The compound interest is given by

I_c = F – P =11,050.79 – 10,000 = P1,050.79.

Example 4. Cris borrows P50,000 and promises to pay the principal and interest at 12% compounded monthly. How much must he repay after 6 years?

Given: P = P50,000 $i^{(12)}$ = 0.12 t = 6 m = 12

Find: F

Solution.

You may also use the other formula to compute for the maturity value

$$F = P\left(1 + \frac{i^{(12)}}{m}\right)^{mt}$$

$$F = (50{,}000)\left(1 + \frac{0.12}{12}\right)^{(12)(6)}$$

$$F = (50{,}000)(1.01)^{72}$$

F = P102,354.97

Thus, Cris must pay P102,354.97 after 6 years.

Present Value P at Compound Interest

$$P = \frac{F}{\left(1 + \frac{i^{(m)}}{m}\right)^{mt}}$$

where F = maturity (future) value
P = principal
$i^{(m)}$ = nominal rate of interest (annual rate)
m = frequency of conversion
t = term/ time in years

Example 5. Find the present value of P50,000 due in 4 years if money is invested at 12% compounded semi-annually.

Given: F = 50,000 t = 4 $i^{(2)}$ = 0.12

Find: P

Solution.

First, compute for the interest rate per conversion period given by

$$j = \frac{i^{(2)}}{m} = \frac{0.12}{2} = 0.06.$$

The total number of conversion periods is n = tm = (4)(2) = 8.

The present value can be computed by substituting these values in the formula

$$P = \frac{F}{(1+j)^n}.$$

Thus,

$$P = \frac{50000}{(1+0.06)^8} = \frac{50000}{(1.06)^8} = P31,370.62.$$

Example 6. What is the present value of P25,000 due in 2 years and 6 months if money is worth 10% compounded quarterly?

Given: F = 25,000 t = 2½ years $i^{(4)} = 0.10$

Find: P

Solution.

The interest rate per conversion period given by

$$j = \frac{i^{(4)}}{m} = \frac{0.10}{4} = 0.025,$$

and the total number of conversion periods is

$$n = tm = (2\ ½)(4) = 10.$$

The present value can be computed by substituting these values in the formula

$$P = \frac{F}{(1+j)^n}.$$

Thus,

$$P = \frac{25,000}{(1+0.025)^{10}} = \frac{25,000}{(1.025)^{10}} = P19,529.96.$$

Solved Examples

A. Debbie wants to compare the simple interest to compound interest on a P 60,000 investment.
   1. Find the simple interest if funds earn 8% simple interest for 1 year.
   2. Find the interest if funds earn 8% compounded annually for 1 year
   3. Find the interest if funds earn 8% compounded semi-annually for 1 year.
   4. Find the interest if funds earn 8% compounded quarterly for 1 year.

5. Which is the best investment? Why?

Given: P = 60,000 $\quad$ r = $i^{(m)}$ = 0.08 t = 1 year

1. I_s = Prt = (60,000)(0.08)(1) = 4,800
2. F = $P(1+r)^t$ = $(60,000)(1+0.08)^1$ = 64,800

   I_c = F – P = 64,800 – 60,000 = 4.800

3. F = $P\left(1+\frac{i^{(2)}}{m}\right)^{tm} = (60{,}000)\left(1+\frac{0.08}{2}\right)^{(1)(2)}$ = 64,896

   I_c = F – P = 64,896 – 60,000 = 4.896

4. F = $P\left(1+\frac{i^{(4)}}{m}\right)^{tm} = (60{,}000)\left(1+\frac{0.08}{4}\right)^{(1)(4)}$ = 64,945.93

   I_c = F – P = 64,945.93 – 60,000 = 4.945.93

5. The investment that yields the highest interest is the one that earns 8% compounded quarterly.

B. James aims to accumulate 1 million pesos in 12 years. Which investment will require the smallest present value?

6. 8% simple interest
7. 8% compounded annually
8. 8% compounded semi-annually
9. 8% compounded quarterly
10. 8% compounded monthly

Given: F = 1,000,000 $\quad$ t = 12 years $\quad$ r=$i^{(m)}$ = 0.08

6. P = $\frac{F}{1+\mathrm{rt}}$

   P = $\frac{1{,}000{,}000}{1+(0.08)(12)}$ = 510,204.08

7. P = $\frac{F}{(1+r)^t}$

   P = $\frac{1{,}000{,}000}{(1+0.08)^{12}}$ = 397,113.76

8. P = $\frac{F}{\left(1+\frac{i^{(2)}}{m}\right)^{mt}}$

   P= $\frac{1{,}000{,}000}{\left(1+\frac{0.08}{2}\right)^{(2)(12)}}$ = 390,121.47

9. P = $\frac{F}{\left(1+\frac{i^{(4)}}{m}\right)^{mt}}$

   P= $\frac{1{,}000{,}000}{\left(1+\frac{0.08}{4}\right)^{(4)(12)}}$ = 386,537.61

10. P = $\frac{F}{\left(1+\frac{i^{(12)}}{m}\right)^{mt}}$

    P= $\frac{1{,}000{,}000}{\left(1+\frac{0.08}{12}\right)^{(12)(12)}}$ = 384,114.67

    (smallest present value)

Lesson 26 Supplementary Exercises

A. Complete the table by computing the interest rate per period and total number of conversion periods.

Nominal Rate $i^{(m)}$	**Interest Compounded**	**Frequency of conversion (m)**	**Interest rate for each conversion period**
10%	Semi-annually	(1)	(2)
(3)	Quarterly	(4)	0.015
12%	Monthly		(5)

B. Complete the table by computing for the maturity values, compound interests and present values.

Present Value	Nominal Rate $i^{(m)}$	Interest compounded	Interest rate per period	Time in Years	Total number of conversions	Compound Interest	Maturity value
20,000	6%	Semi-annually	(6)	8	(7)	(8)	(9)
(10)	10%	quarterly	2.5%	6	(12)		100,000

C. Solve the following problems on compound interest.

13. Cian lends P45,000 for 3 years at 5% compounded semi-annually. Find the future value and interest of this amount.
14. Tenten deposited P10,000 in bank which gives 1% compounded quarterly and let it stay there for 5 years. Find the maturity value and interest.
15. How much should you set aside and invest in a fund earning 9% compounded quarterly if you want to accumulate P200,000 in 3 years and 3 months?
16. How much should you deposit in a bank paying 2% compounded quarterly to accumulate an amount of P80,000 in 5 years and 9 months?
17. Miko has P250,000 to invest at 6% compounded monthly. Find the maturity value if he invests for (a) 2 years? (b) 12 years? (c) How much is the additional interest earned due to the longer time.
18. Yohan sold his car and invested P300,000 at 8.5% compounded quarterly. Find the maturity value if he invests for (a) 3 years? (b) 6 years? (c) How much is the additional interest earned due to the longer time.
19. Maryam is planning to invest P150,000. Bank A is offering 7.5% compounded semi-annually while Bank B is offering 7% compounded monthly. If she plans to invest this amount for 5 years, in which bank should she invest?
20. Yani has a choice to make short term investments for her excess cash P60,000. She can invest at 6% compounded quarterly for 6 months or (b) 5% compounded semi-annually fo 1 year. Which is larger?

Lesson 27: Finding Interest Rate and Time in Compound Interest

Learning Outcome(s): At the end of the lesson, the learner is able to solve problems involving rate of interest and time in compound interest

Lesson Outline:

1. Interest and time in compound interest
2. Equivalent interest rate

Finding the Number of Periods n, for Compounded Interest:

Using the formula for maturity value F, present value P, and interest rate j,

$$F = (1 + j)^n,$$

then

$$\log F = \log(1 + j)^n = n \log(1 + j).$$

Thus,

$$n = \frac{\log F}{m \log(1+j)}.$$

Note that n must be an integer. Some rounding off may be necessary.

Example 1. How long will it take P3,000 pesos to accumulate to P3,500 in a bank savings account at 0.25% compounded monthly?

Given: P = 3,000   F = 3,500 $i^{(12)}$ = 0.25% = 0.0025

m = 12   j = $\frac{i^{(12)}}{m} = \frac{0.0025}{12}$

Find: n and t

Solution.

Substituting the given values in the maturity value formula

$$F = P(1+j)^n$$

results to

$$3500 = 3000\left(1 + \frac{0.0025}{12}\right)^n$$

$$\frac{3500}{3000} = \left(1 + \frac{0.0025}{12}\right)^n$$

To solve for n, take the logarithms of both sides.

$$\log \frac{3500}{3000} = \log\left(1 + \frac{0.0025}{12}\right)^n$$

$$\log\frac{3500}{3000} = n\log\left(1+\frac{0.0025}{12}\right)$$

$$n = \frac{\log\frac{3500}{3000}}{\log\left(1+\frac{0.0025}{12}\right)} = 740.00$$

Thus, payments must be made for 740 months, or $t = \frac{n}{m} = \frac{740}{12} = 61.67$ years.

Example 2. How long will it take P1,000 to earn P300 if the interest is 12% compounded semi-annually?

Given: F = 1,300 $\quad i^{(2)} = 0.12 \quad m = 2 \quad j = \frac{i^{(2)}}{m} = \frac{0.12}{2} = 0.06$

Find: n and t

Solution.

$$F = P(1+j)^n$$

$$1{,}300 = 1{,}000(1 + 0.06)^n$$

$$1.3 = (1.06)^n$$

$$\log(1.3) = \log(1.06)^n$$

$$\log(1.3) = n\log(1.06)$$

$$n = \frac{\log 1.3}{\log(1.06)} \approx 4.503 \text{ periods.}$$

Because interest is earned only at the end of the period, then 5 six-month periods are needed so that the interest can reach P300.

Thus, n = 5, and $t = \frac{n}{m} = \frac{5}{2} = 2.5$ years

It will take 2.5 years for P1,000 to earn P300.

Finding the Interest Rate, j, per Conversion Period

Using the formula maturity value F, present value P, and interest rate j,

$$F = (1+j)^n,$$

then $\sqrt[n]{F} = 1 + j$

Thus, $j = \sqrt[n]{F} - 1$.

Using $j = \frac{i^{(m)}}{m}$, then $i^{(m)} = mj$.

Example 3. At what nominal rate compounded semi-annually will P10,000 accumulate to P15,000 in 10 years?

Given: F = 15,000 P = 10,000 t = 10 m = 2 n = mt = (2)(10) = 20

Find: $i^{(2)}$

Solution.

$$F = P(1+j)^n$$

$$15{,}000 = 10{,}000(1 + j)^{20}$$

$$\frac{15{,}000}{10{,}000} = (1 + j)^{20}$$

$$1.5 = (1 + j)^{20}$$

$$(1.5)^{1/20} = 1 + j$$

$$(1.5)^{1/20} - 1 = j$$

$$j = 0.0205$$

The interest rate per conversion period is 2.05%.

The nominal rate (annual rate of interest) can be computed by

$$j = \frac{i^{(m)}}{m}$$

$$0.0205 = \frac{i^{(2)}}{2}$$

$$i^{(2)} = (0.0205)(2)$$

$$i^{(2)} = 0.0410 \text{ or } 4.10\%$$

Hence, the nominal rate is 4.10%.

Example 4. At what interest rate compounded quarterly will money double itself in 10 years?

Given: F = 2P t = 10 years m = 4 n = mt = (4)(10) = 40

Find: $i^{(4)}$

Solution.

$$F = P(1+j)^n$$

$$2P = P(1+j)^n$$

$$2 = (1 + i)^{40}$$

$$(2)^{1/40} = 1 + j$$

$$(2)^{1/40} - 1 = j$$

$j = 0.0175$ or 1.75%

The interest rate in each conversion period is 1.75%.

The nominal rate can be computed by

$$j = \frac{i^{(4)}}{m}$$

$$0.0175 = \frac{i^{(4)}}{4}$$

$$i^{(4)} = (0.0175)(4)$$

$$i^{(4)} = 0.070 \text{ or } 7.00\%$$

Therefore, the nominal rate that will double an amount of money compounded quarterly in 10 years is 7.0%.

Definition of terms:

Equivalent rates – two annual rates with different conversion periods that will earn the same maturity value for the same time/term

Nominal rate – annual interest rate (may be compounded more than once a year)

Effective rate– rate when compounded annually will give the same compound each year with the nominal rate; denoted by $i^{(1)}$

Example 5. What effective rate is equivalent to 10% compounded quarterly?

Given: $i^{(4)} = 0.10$

$m = 4$

Find: effective rate $i^{(1)}$

Solution.

Since the equivalent rates yield the same maturity value, then

$$F_1 = F_2$$

$$P(1+i^{(1)})^t = P\left(1+\frac{i^{(4)}}{m}\right)^{mt}$$

Dividing both sides by P results to

$$(1+i^{(1)})^t = \left(1+\frac{i^{(4)}}{m}\right)^{mt}$$

Raise both sides to 1/t to obtain

$$(1+i^{(i)}) = \left(1+\frac{0.10}{4}\right)^4$$

$$i^{(1)} = (1.025)^4 - 1 = 0.103813 \text{ or } 10.3813\%$$

Hence, the effective rate equivalent to 10% compounded quarterly is 10.3813%.

Number of Decimal Places
As you will see in Lesson 29, when solving for an equivalent rate j, it is important to make it very precise. Thus ,when solving for an equivalent rate, say $j = (1.025)^4 - 1$ in in Example 5, six or more decimal places will be used.

Example 6. Complete the table by computing for the rates equivalent to the following nominal rates.Round off your answer to six decimal places.

Given Interest Rate	***Equivalent Interest Rate***
12% compounded monthly	____ compounded annually
8 % compounded semi-annually	____ compounded quarterly
12% compounded monthly	____ compounded semi-annually

Solution.

The maturity values accumulated by these interest rates at any time t (in particular, at t = 1) must be equal. That is, $F_1 = F_2$ for any t, including when t = 1.

(1) Given: equal P; equal t

12% compounded monthly	$i^{(12)} = 0.12$	m = 12	P	t
___ compounded annually	$i^{(1)} = ?$	m = 1	P	t

Let F_1 be the future value when interest is compounded annually, and F_2 be the future value when interest is 12% compounded monthly.

$$F_1 = F_2$$

$$P\left(1 + \frac{i^{(1)}}{1}\right)^{(1)t} = P\left(1 + \frac{i^{(12)}}{12}\right)^{12t}$$

$$\left(1 + \frac{i^{(1)}}{1}\right) = \left(1 + \frac{0.12}{12}\right)^{12}$$

$$i^{(i)} = (1.01)^{12} - 1$$

$$i^{(1)} = 0.126825 = 12.6825\%$$

Answer: 12.6825% compounded annually

(2) Given: equal P; equal t

8% compounded semi-annually	$i^{(2)} = 0.08$	m = 2	P	t
___ compounded quarterly	$i^{(4)} = ?$	m = 4	P	t

Let F_1 be the future value when interest is compounded quarterly, and F_2 be the future value when interest is 8% compounded semi-annually.

$F_1 = F_2$

$P(1 + \frac{i^4}{4})^{(4)t} = P(1 + \frac{i^{(2)}}{2})^{(2)t}$

$(1 + \frac{i^{(4)}}{4})^4 = (1 + \frac{0.08}{2})^2$

$(1 + \frac{i^{(4)}}{4})^4 = (1.04)^2$

$1 + \frac{i^{(4)}}{4} = [(1.04)^2]^{(1/4)}$

$1 + \frac{i^{(4)}}{4} = (1.04)^{1/2}$

$1 + \frac{i^{(4)}}{4} = 1.019804$

$\frac{i^{(4)}}{4} = 1.019804 - 1$

$\frac{i^{(4)}}{4} = 0.019804$

$i^{(4)} = (0.019804)(4)$

$i^{(4)} = 0.079216$ or 7.9216%

Answer: 7.9216% compounded quarterly

(3) Given: equal P; equal t

12% compounded monthly	$i^{(12)} = 0.12$	m = 12	P	t
___ compounded semi-annually	$i^{(2)} = ?$	m = 2	P	t

Let F_1 be the future value when interest is compounded semi-annually, and F_2 be the future value when interest is 12% compounded monthly.

$F_1 = F_2$

$P(1 + \frac{i^{(2)}}{2})^{(2)t} = P(1 + \frac{i^{(12)}}{12})^{(12)t}$

$(1 + \frac{i^{(2)}}{2})^2 = (1 + \frac{0.12}{12})^{12}$

$(1 + \frac{i^{(2)}}{2})^2 = (1.01)^{12}$

$1 + \frac{i^{(2)}}{2} = [(1.01)^{12}]^{(1/2)}$

$1 + \frac{i^{(2)}}{2} = (1.01)^6$

$1 + \frac{i^{(2)}}{2} = 1.061520$

$\frac{i^{(2)}}{2} = 1.061520 – 1$

$\frac{i^{(2)}}{2} = 0.061520$

$i^{(2)} = (0.061520)(2)$

$i^{(2)} = 0.12304$ or 12.304%

Answer: 12.304% compounded semi-annually

Solved Examples

1. How long will a principal earn 50% of this amount at 6% compounded quarterly?

Given: $I_c = 0.5P$ $F = 1.5P$ $m = 4$ $i^{(4)} = 0.06$ $j = \frac{i^{(4)}}{m} = \frac{0.06}{4} = 0.015$

Find: t

Solution. $F = P(1+j)^n$

$1.5P = P(1 + 0.015)^n$

$1.5 = (1.015)$

$\log(1.5) = \log(1.015)^n$

$\log(1.5) = n \log(1.015)$

$n = \frac{\log 1.5}{\log (1.015)} = 27.23$ periods

Answer: $t = \frac{n}{m} = \frac{27.23}{4} = 6.81$ years

2. Ethan must pay P15,500 to pay an obligation of P12,000 at 6% compounded monthly. When should this payment be given?

Given: $P = 12{,}000$ $F = 15{,}500$ $i^{(12)} = 6\% = 0.06$ $m = 12$

$j = \frac{i^{(12)}}{m} = \frac{0.06}{12} = 0.005$

Find: t

Solution.

$$F = P(1+j)^n$$

$$15{,}500 = 12{,}000(1+0.005)^n$$

$$\log\left(\frac{15{,}500}{12{,}000}\right) = n\log(1.005)$$

$$n = \frac{\log(12.917)}{\log(1.005)} = 51.3145 \text{ periods}$$

Answer: $t = \frac{n}{m} = \frac{51.3145}{12} = 4.28$ years

3. Shirl is planning to invest P20,000. At what rate compounded semi-annually will accumulate her money to P25,000 in 3 years?

Given: P = 20,000 F = 25,000 t = 3 years m = 2 n = mt = (2)(3) = 6

Find: $i^{(2)}$

Solution. $F = P(1+j)^n$

$$25{,}000 = (20{,}000)(1+j)^n$$

$$1.25 = (1+j)^6$$

$$(1.25)^{1/6} = 1 + j$$

$$(1.25)^{1/6} - 1 = j$$

$$j = 0.0379 \quad \text{or } 3.79\%\%$$

The interest rate in each conversion period is 3.79%.

The nominal rate can be computed by

$$j = \frac{i^{(12)}}{m}$$

$$0.0379 = \frac{i^{(2)}}{2}$$

$$i^{(2)} = (0.0370)(2)$$

Answer: $^{(2)} = 0.0758$ or 7.58%

4. What nominal rate compounded monthly is equivalent to 12% compounded annually? Round off your answer to six decimal places.

Given: $i^{(1)} = 0.12$ m =1

Find $i^{(12)}$

Solution. $F_1 = F_2$

$$P\left(1+\frac{i^{(12)}}{m}\right)^{mt} = P(1+j)^t$$

$$\left(1+\frac{i^{(12)}}{12}\right)^{12t} = (1+i^{(1)})^t$$

$$\left(1+\frac{i^{(12)}}{12}\right)^{12} = (1+0.12)$$

$$\frac{i^{(12)}}{12} = (1.12)^{(1/12)} - 1 = 0.009489$$

$$i^{(12)} = 12(0.009489) = 0.113868 \text{ or } 11.3868\%$$

Answer: The nominal rate compounded monthly equivalent to 12% compounded annually is 11.3868%.

5. What simple interest rate is equivalent to 10% compounded semi-annually at the end of year 1?

Given: $i^{(2)} = 0.10$ $m = 2$ $t = 1$

Find: r

Solution.

Simple Interest = Compound Interest

$$F_s = F_c$$

$$P(1 + r_s t) = P\left(1+\frac{i^{(2)}}{m}\right)^{mt}$$

$$(1 + r_s t) = \left(1+\frac{i^{(2)}}{2}\right)^{2t}$$

Substitute t = 1 to obtain:

$$(1 + r_s) = \left(1+\frac{i^{(2)}}{2}\right)^{2}$$

$$(1 + r_s) = \left(1+\frac{0.10}{2}\right)^{2}$$

$(1 + r_s) = (1.05)^2$

$r_s = (1.05)^2 - 1$

Answer: $r_s = 0.1025$ or 10.2

Lesson 27 Supplementary Exercises

A. Complete the table by computing for unknown values. In numbers 3, 6, and 9, round off your answer to six decimal places.

Nominal Rate	***Interest compounded***	***Frequency of Conversion Periods***	***Interest Rate per Period***	***Equivalent Nominal Rate***	***Rate per period, based on rate and period from previous column***
10%	quarterly	(1)	(2)	___(3)_ compounded semi-annually	(4)
6%	semi-annually	2	(5)	__(6)___ compounded monthly	(7)
12%	monthly	12	(8)	__(9)___ compounded quarterly	(10)

B. Complete the table by finding the unknown time and rate.

Principal	Nominal Rate	Interest compounded	Frequency of Conversions	Interest rate per period	Time in Years	Number of conversions	Compound Interest	Maturity value
8,000	(11)	quarterly	(12)	(13)	6	(14)	(15)	8,800
75,000	11%	Semi-annually	(16)	(17)	(18)	(19)	10,000	(20)

C. Solve the following problems:

21. Jun invested an amount of P 100,000 where he obtained an interest of 16,000at the end of 2 ½ years. At what nominal rate compounded semi-annually was it invested?
22. Jen invested an amount of P 400,000 at 5% compounded quarterly. How long should she let the investment stay if she wants to earn P 50,000?
23. Mr. Retanan was given a loan at 10% compounded monthly. When should he pay it so that it will just earn only 10% of the amount borrowed?
24. At what interest rate compounded quarterly should an amount be invested if the interest earned is 20% of the invested amount for 5 years?
25. What simple interest rate is equivalent to 1% compounded quarterly?

Lesson 28: Simple Annuity

Learning Outcome(s): At the end of the lesson, the learner is able to illustrate simple and general annuities, distinguish between simple and general annuities, find the future and present values of simple annuities, computes the periodic payment of a simple annuity

Lesson Outline:

1. Definition of Terms
2. Time Diagrams
3. Future Value of a Simple Annuity
4. Present Value of a Simple Annuity
5. Periodic Payment of a Simple Annuity

Definition of Terms:

ANNUITY– a sequence of payments made at equal(fixed) intervals or periods of time

Annuities may be classified in different ways, as follows.

	Annuities	
According to payment interval and interest period	**Simple Annuity**- an annuity where the payment interval is the same as the interest period	**General Annuity -** an annuity where the payment interval is not the same as the interest period
According to time of payment	**Ordinary Annuity (or Annuity Immediate)** – a type of annuity in which the payments are made at the end of each payment interval	**Annuity Due** – a type of annuity in which the payments are made at beginning of each payment interval
According to duration	**Annuity Certain–** an annuity in which payments begin and end at definite times	**Contingent Annuity –** an annuity in which the payments extend over an indefinite (or indeterminate) length of time

Note: Grade 11 will focus on Ordinary Annuities (not Annuity Due), and on Annuity Certain (not Contingent Annuities). Simple Annuities are discussed in this lesson, and General Annuities are discussed on Lesson 29.

Term of an annuity, t – time between the first payment interval and last payment interval

Regular or Periodic payment, R – the amount of each payment

Amount (Future Value) of an annuity, F – sum of future values of all the payments to be made during the entire term of the annuity

Present value of an annuity, P – sum of present values of all the payments to be made during the entire term of the annuity

Annuities may be illustrated using a time diagram. The time diagram for an ordinary annuity (i.e., payments are made at the end of the year) is given below.

Time Diagram for an n-PaymentOrdinary Annuity

	R	R	R	R	R	…	R
0	1	2	3	4	5		n

Example 1. Suppose Mrs. Remoto would like to save P3,000 every month in a fund that gives 9% compounded monthly. How much is the amount or future value of her savings after 6 months?

Given: periodic payment R = P3,000

term t = 6 months

interest rate per annum $i^{(12)} = 0.09$

number of conversions per year m = 12

interest rate per period $j = \frac{0.09}{12} = 0.0075$

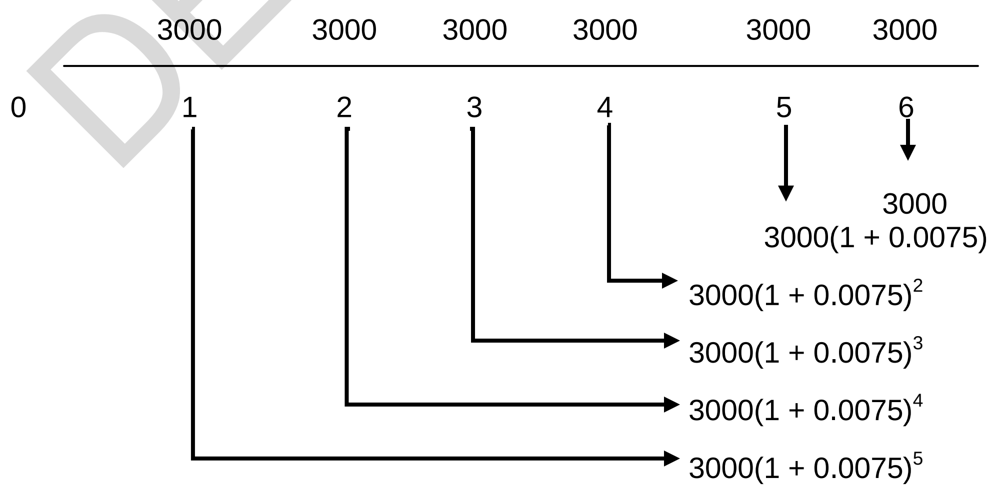
Find: amount (future value) at the end of the term, F

Solution.

(1) Illustrate the cash flow in a time diagram.

	3000	3000	3000	3000	3000	3000
0	1	2	3	4	5	6

(2) Find the future value of all the payments at the end of term (t = 6).



(3) Add all the future values obtained from the previous step.

$3000 = 3000$

$3000(1 + 0.0075) = 3022.5$

$3000(1 + 0.0075)^2 = 3045.169$

$3000(1 + 0.0075)^3 = 3068.008$

$3000(1 + 0.0075)^4 = 3091.018$

$3000(1 + 0.0075)^5 = 3114.20$

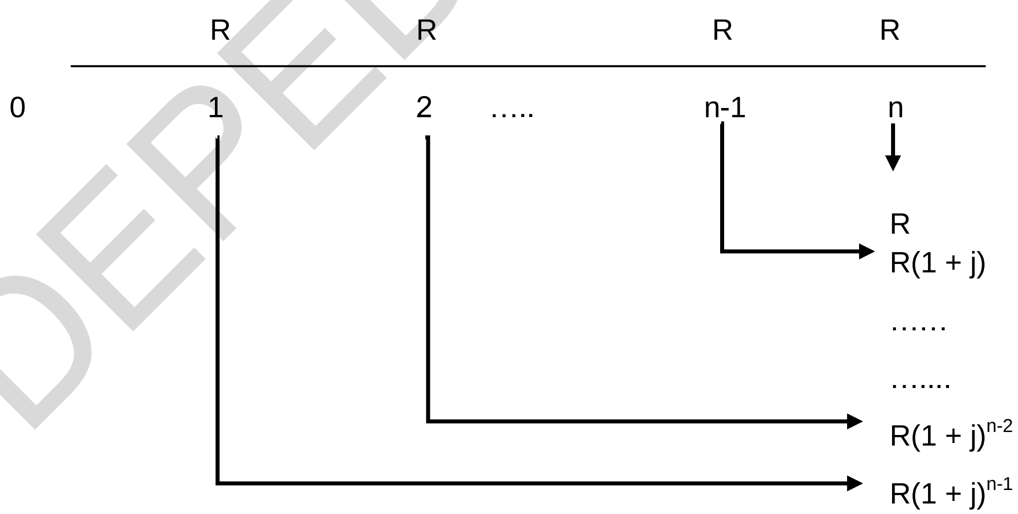
$F = 18340.89$

Thus, the amount of this annuity is P18,340.89.

Amount (Future Value) of an Ordinary Annuity:

The derivation of the formula in finding the amount of an ordinary annuity is similar to the solution of Example 1.

Illustrate the cash flow in a time diagram.



$$F = R + R(1 + j) + R(1 + j)^2 + \ldots + R(1 + j)^{n-2} + R(1 + j)^{n-1} \qquad (1)$$

Multiply both sides by (1 + j) to get

$$F(1+j) = R(1+j) + R(1 + j)^2 + R(1+j)^3 + \ldots + R(1+j)^{n-1} + R(1+j)^n \qquad (2)$$

From Equation (1), subtract Equation (2) to produce

$F(1+j) – F = R(1+j)^n - R$

$F[(1+j) - 1] = R[(1+j)^n – 1]$

$F(j) = R [(1+j)^n-1]$

$$F = R\ \frac{(1+j)^n - 1}{j}$$

The expression $\frac{(1+j)^n - 1}{j}$ is usually denoted by the symbol $s_{\overline{n}|}$ read as "s angle n".

Alternate Solution to Example 1:

> **Amount (Future Value) of ordinary annuity:**
>
> The future value F of an ordinary annuity is given by
>
> $$F = R\ \frac{(1+j)^n - 1}{j}$$
>
> where R is the regular payment;
> j is the interest rate per period;
> n is the number of payments

$$F = R\ \frac{(1+j)^n - 1}{j}$$

$$F = (3000)\ \frac{(1 + 0.0075)^6 - 1}{0.0075}$$

F = 18,340.89.

Example 2. In order to save for her high school graduation, Marie decided to save P200 at the end of each month. If the bank pays 0.250% compounded monthly, how much will her money be at the end of 6 years?

Given: R = 200

m = 12

$i^{(12)} = 0.250\% = 0.0025$

$j = \frac{0.0025}{12} = 0.000208\overline{3}$

t = 6 years

$n = tm = (6)(12) = 72$ periods

Find: F

Solution.

$$F = R\ \frac{(1+j)^n - 1}{j}$$

$$F = (200)\ \frac{(1 + 0.000208\overline{3})^{72} - 1}{0.000208\overline{3}}$$

$F = 14,507.85$

Hence, Marie will be able to save P14,507.85 for her graduation

Example 3: (Recall the problem in Example 1.) Suppose Mrs. Remoto would like to know the present value of her monthly deposit of P3,000 when interest is 9% compounded monthly. How much is the present value of her savings at the end of 6 months?

Given: periodic payment $R = 3,000$

term $t = 6$ months

interest rate per annum $i^{(12)} = 0.09$

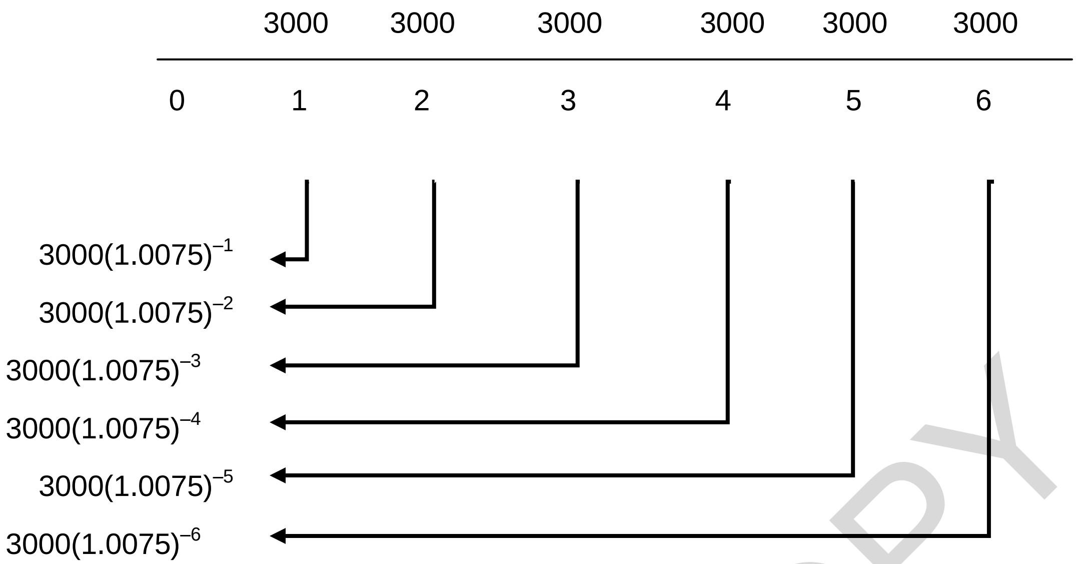
number of conversions per year $m = 12$

interest rate per period $j = \frac{0.09}{12} = 0.0075$

Find: Present value P

(1) Discount the payment of each period to the beginning of the term. That is, find the present value of each payment. Recall the formula

$$P = \frac{F}{\left(1 + \frac{i^{(m)}}{m}\right)^{mt}} = \frac{3,000}{(1.0075)^t} = 3,000(1.0075)^{-t}.$$



(2) Add the discounted payments to get the present value.

$3000(1.0075)^{-1}$	=2977.667
$3000(1.0075)^{-2}$	= 2955.501
$3000(1.0075)^{-3}$	= 2933.50
$3000(1.0075)^{-4}$	= 2911.663
$3000(1.0075)^{-5}$	= 2889.988
$3000(1.0075)^{-6}$	= 2868.474
	P =17536.79

Thus, the cost of the TV set at the beginning of the term is P17,536.79.

Alternate Solution to Example 3:

Since we already know from Example 1 that the accumulated amount at the end of 6 months is P18,340.89, then we can simply get the present value of this amount using the formula

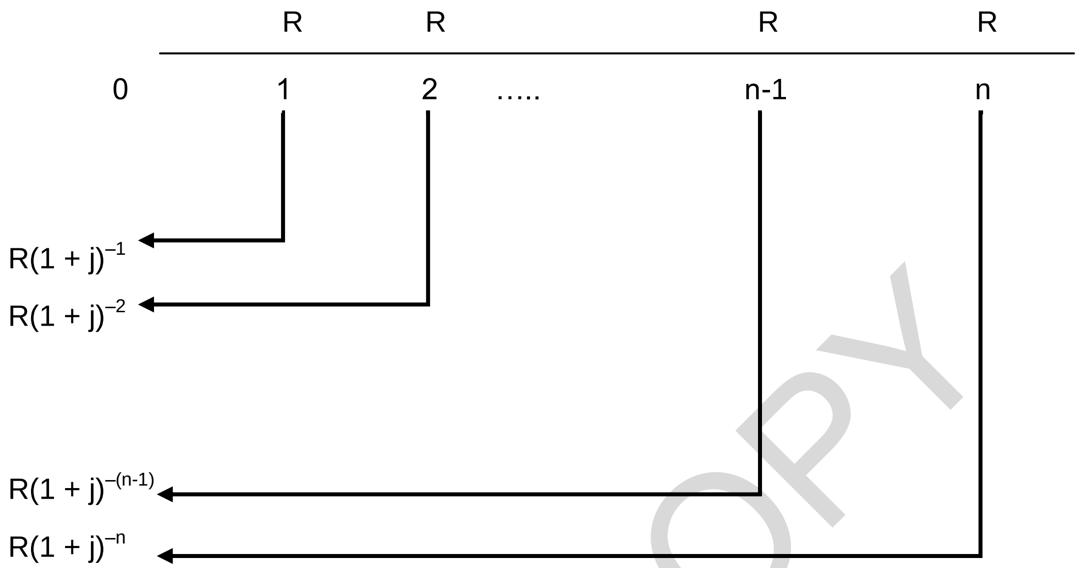
$$P = \frac{F}{(1+j)^n} = \frac{F}{\left(1+\frac{i^{(m)}}{m}\right)^{tm}} = \frac{18340.89}{\left(1+\frac{.09}{12}\right)^6} = 17536.79.$$

Present Value of an Ordinary Annuity:

The derivation of the formula in finding the amount of an ordinary annuity is similar to the solution of Example 3.

*** Discount or get the value of each payment at the beginning of the term and then add to get the present value of an ordinary annuity. Use the formula

$$P = \frac{F}{(1+j)^n} = F(1+j)^{-n}$$



$$P = R(1+j)^{-1} + R(1+j)^{-2} + \cdots + R(1+j)^{-(n-1)} + R(1+j)^{-n}$$

$$P = \frac{R}{(1+j)^1} + \frac{R}{(1+j)^2} + \frac{R}{(1+j)^3} \ldots + \frac{R}{(1+j)^{n-1}} + \frac{R}{(1+j)^n} \qquad (1)$$

Multiply both sides by $\frac{1}{1+j}$ to get

$$\frac{P}{1+j} = \frac{R}{(1+j)^2} + \frac{R}{(1+j)^3} + \cdots + \frac{R}{(1+j)^n} + \frac{R}{(1+j)^{n+1}} \qquad (2)$$

From Equation (1), subtract Equation (2) to produce

$$P - P\frac{1}{1+j} = \frac{R}{1+j} - \frac{R}{(1+j)^{n+1}}$$

$$P\left(1 - \frac{1}{1+j}\right) = \frac{R}{1+j}\left(1 - \frac{1}{(1+j)^n}\right)$$

$$P\left(\frac{1+j-1}{1+j}\right) = \frac{R}{1+j}\left(1-(1+j)^{-n}\right)$$

$$P\left(\frac{j}{1+j}\right) = \frac{R}{1+j}\left(1-(1+j)^{-n}\right)$$

$$Pj = R\left(1-(1+j)^{-n}\right)$$

$$P = R\frac{1-(1+j)^{-n}}{j}$$

The expression $\dfrac{1-(1+j)^{-n}}{j}$ is usually denoted by the symbol $a_{\overline{n}|}$ read as “a angle n”.

Hence, the present value P of an ordinary annuity can be written as

$$P = R\,a_{\overline{n}|} = R\,\frac{1-(1+j)^{-n}}{j}.$$

<u>Alternative Derivation</u>

The future value of an ordinary annuity was given earlier by

$$F = R\,\frac{(1+j)^n - 1}{j}$$

Present Value of an Ordinary Annuity

The present value of an ordinary annuity is given by

$$P = R\,\frac{1-(1+j)^{-n}}{j}$$

where R is the regular payment;
j is the interest rate per period;
n is the number of payments

To get the present value of this amount, we use the formula $P = \dfrac{F}{(1+j)^n}$ and obtain

$$P = \frac{F}{(1+j)^n} = \frac{R\,\frac{(1+j)^n-1}{j}}{(1+j)^n} = R\,\frac{(1+j)^n-1}{j}(1+j)^{-n} = R\,\frac{1-(1+j)^{-n}}{j}.$$

<u>Alternate Solution to Example 3:</u>

$$P = R\,\frac{1-(1+j)^{-n}}{j}$$

$$P = (3000)\,\frac{1-(1+0.0075)^{-6}}{0.0075}$$

P =17,536.79.

The **cash value** or **cash price** of a purchase is equal to the down payment (if there is any) plus the present value of the installment payments.

Example 4. Mr. Ribaya paid P200,000 as down payment for a car. The remaining amount is to be settled by paying P16,200 at the end of each month for 5 years. If interest is 10.5% compounded monthly, what is the cash price of his car?

Given: down payment = 200,000
R = 16,200
$i^{(12)} = 0.105$
m = 12
$j = \frac{0.105}{12} = 0.00875$
t = 5 years
n = mt = (12)(5) = 60 periods

Find: cash value or cash price of the car

Solution.

The time diagram for the installment payments is given by:

P=?

	16200	16200	16200	…	16200
0	1	2	3	…	60

The present value of this ordinary annuity is given by

$$P = R \frac{1-(1+j)^{-n}}{j}$$

$$P = (16200)\frac{1-(1+0.00875)^{-60}}{0.00875}$$

P = 753,702.20

Cash value = Down payment + present value
= 200,000 + 753,702.20
Cash Value = P953,702.20

The cash price of the car is P953,702.20

Periodic payment R of an Annuity:

Periodic payment R can also be solved using the formula for amount F or present value P of an annuity.

$$F = R\left(\frac{(1+j)^n - 1}{j}\right) \implies R = F/\left(\frac{(1+j)^n - 1}{j}\right)$$

$$F = R\left(\frac{1-(1+j)^{-n}}{j}\right) \implies R = F/\left(\frac{1-(1+j)^{-n}}{j}\right)$$

where R is the regular payment;
P is the present value of an annuity
F is the future value of an annuity
j is the interest rate per period;
n is the number of payments

Example 5. Paolo borrowed P 100 000. He agrees to pay the principal plus interest by paying an equal amount of money each year for 3 years. What should be his annual payment if interest is 8% compounded annually?

Given: P = 100 000
$i^{(1)} = 0.08$
m = 1
j = 0.08
t = 3 years
n = mt = (1)(3) = 3 periods
Find: periodic payment R

Solution. The cash flow of this annuity is illustrated in the time diagram given below.

P=100000

	R = ?	R = ?	R = ?
0	1	2	3

Since $P = R \frac{1-(1+j)^{-n}}{j}$ then

$$R = P / \left[\frac{1-(1+j)^{-n}}{j}\right]$$

$$R = 100000 / \left[\frac{1-(1+0.08)^{-3}}{0.08}\right]$$

R = 38,803.35

Thus, the man should pay P38,803.35 every year for 3 years.

Solved Examples

1. Aling Paring started to deposit P2,000 quarterly in a fund that pays 5.5% compounded quarterly. How much will be in the fund after 6 years?

Given: R = 2,000
m = 4
$i^{(12)}$ = 5.5% = 0.055
$j = \frac{0.055}{4} = 0.01375$
t = 6 years
n = tm = (6)(4)= 24 periods
Find: F

Solution. $F = R\,\frac{(1+j)^n - 1}{j}$

$$F = (2{,}000)\,\frac{(1+0.01375)^{24} - 1}{0.01375}$$

Answer: F = 56,413.75

2. The buyer of a house and lot pays P200,000 cash and P10,000 every month for 20 years. If money is 9% compounded monthly, how much is the cash value of the lot?

Given: down payment = 200,000
R = 10,000
$i^{(12)}$ = 0.09
m = 12
$j = \frac{0.09}{12} = 0.0075$
t = 20 years
n = mt = (12)(20) = 240 periods

Find: Cash value

Solution.

$$P = R\ \frac{1-(1+j)^{-n}}{j}$$

$$P = (10{,}000)\ \frac{1-(1+0.0075)^{-240}}{0.0075}$$

P = 1,111,449.54

Cash Value = Down payment + Present Value

Cash Value = 200,000 + 1,111,449.54

Answer: Cash Value = 1,311,449.54

3. Grace borrowed P150,000 payable in 2 years. To repay the loan, she must pay an amount every month with an interest rate of 6% compounded monthly. How much should he pay every month?

Given: P = 150 000
$i^{(1)} = 0.06$
m = 12
j = 0.005
t = 2 years
n = mt = (12)(2) = 24 periods

Find: periodic payment R

Solution.

$$R = P/\ [\frac{1-(1+j)^{-n}}{j}]$$

$$R = 150000/\ [\frac{1-(1+0.005)^{-24}}{0.005}]$$

Answer: R = 6,6408.09

4. Mr. Ribaya would like to save P500,000 for his son's college education. How much should he deposit in a savings account every 6 months for 12 years if interest is at 1% compounded semi-annually?

Given: F = 500,000
$i^{(2)} = 0.01$
m = 2
j = 0.005
t = 12 years
n = mt = (2)(12) = 24 periods

Find: periodic payment R

Solution.

$$R = F \;/\; \frac{(1+j)^n - 1}{j}$$

$$R = 500{,}000 \;/\; \frac{(1+0.005)^{24} - 1}{0.005}$$

Answer: R = P19,660.31

5. A refrigerator is for sale at P17,999 in cash or on terms, P1,600 each month for the next 12 months. Money is 9% compounded monthly. Which is lower, the cash price or the present value of the installment terms?
   Given:

   Cash Price: P 17,999
   $R = 1{,}600$
   $i^{(12)} = 0.09$
   $m = 12$
   $j = 0.0075$
   $t = 1$ year
   $n = mt = (12)(1) = 12$ periods

   Find: Present Value P

   Solution.

$$P = R\,\frac{1-(1+j)^{-n}}{j}$$

$$P = (1{,}600)\,\frac{1-(1+0.0075)^{-12}}{0.0075}$$

$$P = 18{,}295.86$$

Therefore, buying the refrigerator in cash is lower than paying it in installment term.

Lesson 28 Supplementary Exercises

A. Find the future value F of the following ordinary annuities.

1. Monthly payments of P3,000 for 4 years with interest rate of 3% compounded monthly
2. Quarterly payment of P5,000 for 10 years with interest rate of 2% compounded quarterly
3. Semi-annual payments of P12, 500 with interest rate of 10.5% compounded semi-annually for 6 years
4. Annual payments of P105,000 with interest rate of 12% compounded annually for 5 years
5. Daily payments of P20 for 30 days with interest rate of 20% compounded daily for 1 month

B. Find the present value P of the following ordinary annuities.

6. Monthly payments of P2,000 for 5 years with interest rate of 12% compounded monthly
7. Quarterly payment of P15,000 for 10 years with interest rate of 8% compounded quarterly
8. Semi-annual payments of P20,500 with interest rate of 8.5% compounded semi-annually for 3 years
9. Annual payments of P150,000 with interest rate of 8% compounded annually for 10 years
10. Daily payments of P54 for 30 days with interest rate of 15% compounded daily for one month

C. Find the periodic payments of the following ordinary annuities.

11. Monthly payment of the future value of P50,000 for 1 year with an interest rate of 10% compounded monthly
12. Quarterly payment of an accumulated amount of P80,000 for 2 years with interest rate of 8% compounded quarterly
13. Payment every six months for the present value of P100,000 for 2 years with an interest rate of 12% compounded semi-annually
14. Annual payment of the loan P800,000 for 5 years with an interest rate of 9% compounded annually
15. Monthly installment of an appliance cash prize of P20,000 for 6 months with an interest rate of 6% compounded monthly

D. Solve the following problems.

16. How much is the monthly amortization on an automobile loan of P900,000 to be amortized over a 5-year period at a rate 9.5% compounded monthly?

17. Shirl started to deposit P18,000 semi-annually in a fund that pays 5% compounded semi-annually. How much will be in the fund after 10 years?

18. Kathrina wants to buy a lot which costs 1 million pesos. She plans to give a down payment of 20% of the cost, and the rest will be paid by financing at annual interest rate of 12% for 10 years in equal monthly installments? What will be the monthly payment?

19. Ken is paying P2,500 every 3 months for the amount he borrowed at an interest rate of 8% compounded quarterly. How much did he borrow if he agreed that the loan will be paid in 2 years and 6 months?

20. A store advertises a motorcycle for P3,000 downpayment and P3,000 per month for 15 months. If the interest is 15% compounded monthly, what is the actual value of the motorcycle?

Lesson 29: General Annuity

Learning Outcome(s): At the end of the lesson, the learner is able to find the future and present values of general annuities and compute the periodic payment of a general annuity, and calculate the fair market value of a cash flow stream that includes an annuity.

Lesson Outline:

1. Future Value of a General Annuity
2. Present Value of a General Annuity
3. Fair Market Value of a Cash Flow Stream that Includes and Annuity

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Recall these terms defined in Lesson 28.

General Annuity–an annuity where the length of the payment interval is not the same as the length of the interest compounding period

General Ordinary Annuity – a general annuity in which the periodic payment is made at the end of the payment interval

Examples of General annuity:

1. Monthly installment payment of a car, lot, or house with an interest rate that is compounded annually
2. Paying a debt semi-annually when the interest is compounded monthly

Future and Present Value of a General Ordinary Annuity

The future value F and present value P of a general ordinary annuity is given by

$$F = R\,\frac{(1+j)^{n}-1}{j} \quad \text{and} \quad P = R\,\frac{1-(1+j)^{-n}}{j}$$

where R is the regular payment;
j is the equivalent interest rate per payment interval converted from the interest rate per period; and
n is the number of payments.

The formulas for F and P are same as those in Lesson 28. The **extra step** occurs in finding j: the given interest rate per period must be converted to an equivalent rate per payment interval.

Example 1. Cris started to deposit P1,000 monthly in a fund that pays 6% compounded quarterly. How much will be in the fund after 15 years?

Given: R = 1,000   n = 12(15) = 180 payments $i^{(4)} = 0.06$ m = 4

Find: F

Solution:

The cash flow for this problem is shown in the diagram below.

						F
	1000	1000	1000	….	1000	1000
0	1	2	3	…..	179	180

(1) Convert 6% compounded quarterly to its equivalent interest rate for monthly payment interval.

$F_1 = F_2$

$P\left(1+\frac{i^{(12)}}{12}\right)^{(12)t} = P\left(1+\frac{i^{4}}{4}\right)^{(4)t}$

$\left(1+\frac{i^{(12)}}{12}\right)^{12} = \left(1+\frac{0.06}{4}\right)^{4}$

$\left(1+\frac{i^{(12)}}{12}\right)^{12} = (1.015)^{4}$

$1+\frac{i^{(12)}}{12} = [(1.015)^{4}]^{(1/12)}$

$\frac{i^{(12)}}{12} = (1.015)^{1/3} - 1$

$\frac{i^{(12)}}{12} = 0.00497521 = j$

Thus, the interest rate per monthly payment interval is 0.00497521 or 0.497521%.

(2) Apply the formula in finding the future value of an ordinary annuity using the computed equivalent rate

$$F = R\ \frac{(1+j)^n - 1}{j}$$

$$F = 1000\ \frac{(1+0.00497521)^{180} - 1}{0.00497521}$$

F = 290,082.51

Thus, Cris will have P290,082.51 in the fund after 20 years.

Number of Decimal Places
When solving for an equivalent rate, say $j = (1.015)^{1/3} - 1$ in Example 1, six or more decimal places will be used. If you use fewer or more decimal places, your answers may differ from the answers provided in the text. *You can ignore these discrepancies*, but it is suggested that you use at least six decimal places, or the exact value.

Example 2. A teacher saves P5,000 every 6 months in a bank that pays 0.25% compounded monthly. How much will be her savings after 10 years?

Given: R = 5000

n = 2(10) = 20 payments

$i^{(12)}$ = 0.25% = 0.0025

m = 12

Find: F

Solution.

The cash flow for this problem is shown in the diagram below.

	5000	5000	5000	…	5000	5000
0	1	2	3	…	19	20

(1) Convert 0.25% compounded monthly to its equivalent interest rate for each semi-annual payment interval.

$F_1 = F_2$

$P(1 + \frac{i^2}{2})^{(2)t} = P(1 + \frac{i^{(12)}}{12})^{(12)t}$

$(1 + \frac{i^{(2)}}{2})^2 = (1 + \frac{0.0025}{12})^{12}$

$(1 + \frac{i^{(2)}}{2})^2 = (1.00020833)^{12}$

$1 + \frac{i^{(2)}}{2} = [(1.00020833)^{12}]^{(1/2)}$

$\frac{i^{(2)}}{2} = (1.00020833)^6 - 1$

$\frac{i^{(2)}}{2} = 0.00125063 = j$

Thus, the interest rate per semi-annual payment interval is 0.00125063 or 0.125%.

(2) Apply the formula in finding the future value of an ordinary annuity using the computed equivalent rate

$$F = R \frac{(1 + j)^n - 1}{j}$$

$$F = 5000 \frac{(1 + 0.00125063)^{20} - 1}{0.00125063}$$

F = 101,197.06

Thus, the teacher will be able to save P101,197.06 after 10 years.

Example 3. Ken borrowed an amount of money from Kat. He agrees to pay the principal plus interest by paying P38,973.76 each year for 3 years. How much money did he borrow if interest is 8% compounded quarterly?

Given: R = 38,973.76

$i^{(4)} = 0.08$

m = 4

n = 3 payments

Find: present value P

Solution.

The cash flow for this problem is shown in the diagram below.

P=?

0	1	2	3
	R=38973.76	R=38973.76	R = 38973.76

(1) Convert 8% compounded quarterly to its equivalent interest rate for each payment interval.

$F_1 = F_2$

$P\left(1 + \frac{i^{(1)}}{1}\right)^{(1)t} = P\left(1 + \frac{i^{(4)}}{4}\right)^{4t}$

$\left(1 + \frac{i^{(1)}}{1}\right) = \left(1 + \frac{0.08}{4}\right)^{4}$

$\left(1 + \frac{i^{(1)}}{1}\right) = (1 + 0.02)^{4}$

$\frac{i^{(1)}}{1} = (1.02)^{4} - 1$

$\frac{i^{(1)}}{1} = j = 0.082432 = 8.24\%$

Thus, the interest rate per payment interval is 0.082432 or 8.24%.

(2) Apply the formula in finding the present value of an ordinary annuity using the computed equivalent rate j = 0.082432.

$$P = R\ \frac{1 - (1 + j)^{-n}}{j}$$

$$P = 38973.76\ \frac{1 - (1 + 0.082432)^{-3}}{0.082432}$$

P = 100,000.00

Hence, Ken borrowed P100,000 from Kat.

Example 4. Mrs. Remoto would like to buy a television (TV) set payable for 6 months starting at the end of the month. How much is the cost of the TV set if her monthly payment is P3,000 and interest is 9% compounded semi-annually?

Given: R = 3,000   m = 2

$i^{(2)} = 0.09$ n = 6 payments

Find: cost (present value) at the beginning of the term P

Solution:

$$F_1 = F_2$$

$$P\left(1 + \frac{i^{(12)}}{12}\right)^{(12)t} = P\left(1 + \frac{i^{(2)}}{2}\right)^{(2)t}$$

$$\left(1 + \frac{i^{(12)}}{12}\right)^{12} = \left(1 + \frac{0.09}{2}\right)^{2}$$

$$\left(1 + \frac{i^{(12)}}{12}\right)^{12} = (1+0.045)^{2}$$

$$1 + \frac{i^{(12)}}{12} = [\ (1.045)^{2}]^{(1/12)}\ ]$$

$$\frac{i^{(12)}}{12} = (1.045)^{1/6} - 1$$

$$\frac{i^{(12)}}{12} = 0.00736312 = j$$

Thus, the interest rate per monthly payment interval is 0.00736312 or 0.736312%.

(2) Apply the formula in finding the present value of an ordinary annuity using the computed equivalent rate j = 0.00736312.

$$P = R\ \frac{1 - (1 + j)^{-n}}{j}$$

$$P = 3000\ \frac{1 - (1 + 0.00736312)^{-6}}{0.00736312}$$

P = 17,545.08

Thus, the cost of the TV set is P17,545.08.

A **cash flow** is a term that refers to payments received (cash inflows) or payments or deposits made (cash outflows). Cash inflows can be represented by positive numbers and cash outflows can be represented by negative numbers.

The **fair market value** or **economic value** of a cash flow (payment stream) on a particular date refers to a *single amount* that is equivalent to the value of the payment stream at that date. This particular date is called the **focal date**.

Example 5. Mr. Ribaya received two offers on a lot that he wants to sell. Mr. Ocampo has offered P50,000 and a P1 million lump sum payment 5 years from now. Mr. Cruz has offered P50,000 plus P40,000 every quarter for five years. Compare the fair market values of the two offers if money can earn 5% compounded annually. Which offer has a higher market value?

Given:

Mr. Ocampo's offer	***Mr. Cruz's offer***
P50,000 down payment	P50,000 down payment
P1,000,000 after 5 years	P40,000 every quarter for 5 years

Find: fair market value of each offer

Solution.

We illustrate the cash flows of the two offers using time diagrams.

Mr. Ocampo's offer:

50,000 **1 million**

0 1 2 3 4 5

Mr. Cruz's offer:

50,000
40,000 40,000 40,000 … 40,000

0 1 2 3 … 20

Choose a focal date and determine the values of the two offers at that focal date. For example, the focal date can be the date at the start of the term.

Since the focal date is at t = 0, compute for the present value of each offer.

Mr. Ocampo's offer: Since P50,000 is offered today, then its present value is still PhP 50 000.The present value of P1,000,000 offered 5 years from now is

$$P = F(1 + j)^{-n}$$

$$P = 1,000,000\ (1+0.05)^{-5}$$
$$P = P783,526.20$$

Fair Market Value (FMV) = Down payment + Present Value

= 50,000 + 783,526.20

FMV = P833,526.20

Mr. Cruz's offer: We first compute for the present value of a general annuity with quarterly payments but with annual compounding at 5%.

Solve the equivalent rate, compounded quarterly, of 5% compounded annually.

$F_1 = F_2$

$P(1 + \frac{i^{(4)}}{4})^{(4)(5)} = P(1 + \frac{i^{(1)}}{1})^{(1)(5)}$

$(1 + \frac{i^{(4)}}{4})^{20} = (1 + \frac{0.05}{1})^{5}$

$1 + \frac{i^{(4)}}{4} = (1.05)^{(1/4)}$

$\frac{i^{(4)}}{4} = (1.05)^{(1/4)} - 1$

$\frac{i^{(4)}}{4} = 0.012272$

The present value of an annuity is given by

$$P = R \frac{1 - (1 + j)^{-n}}{j}$$

$$P = 40000 \frac{1 - (1 + 0.012272)^{-20}}{0.012272}$$

P = P705,572.70

Fair Market Value = Downpayment + present value

= 50,000 + 705,572.70

Fair Market Value = P755,572.70

Hence, Mr. Ocampo's offer has a higher market value. The difference between the market values of the two offers at the start of the term is

833 526.20 - 755 572.70 = P77,953.50

Alternate Solution (Focal date at the end of the term):

Mr. Ocampo's offer:

The future value of P1,000,000 at the end of the term at 5% compounded annually is given by

$$F = P(1 + j)^n$$

$$F = 50{,}000(1+0.05)^5$$

$$F = 63{,}814.08$$

The fair market value of this offer at the end of the term is 63,814.08 plus 1 million pesos amounting to P,1,063 814.08.

Mr. Cruz's offer:

The future value of this ordinary general annuity is given by:

$$F = R\frac{(1+j)^n - 1}{j}$$

$$F = 40000\frac{(1+0.012272)^{20} - 1}{0.012272}$$

$$F = 900{,}509.40$$

The future of P50,000 at the end of the term is P63,814.08, which was already determined earlier.

Fair Market Value = 900,509.40 + 63,814.08 = P964,323.48

As expected, Mr. Ocampo's offer still has a higher market value, even if the focal date was chosen to be at the end of the term. The difference between the market values of the two offers at the end of the term is

1,063,814.08 –964,323.48 = P99,490.60.

You can also check that the present value of the difference is the same as the difference computed when the focal date was the start of the term:

$P = 99{,}490.60(1 + 0.05)^{-5}$ = P77,953.49.

Example 6. Which offer has a better market value?

Company A offers P150,000 at the end of 3 years plus P300,000 at the end of 5 years. Company B offers P25,000 at the end of each quarter for the next 5 years. Assume that money is worth 8% compounded annually.

Given:

Company A	Company B
P150,000 at the end of 3 years P300,000 at the end of 5 years	P25,000 at the end of each quarter for the next 5 years

Find: fair market value of each offer

Solution.

(1) Illustrate the cash flows of the two offers using time diagrams.

Company A offer:

			150,000		**300,000**
0	1	2	3	4	5

Company B offer:

	25,000	25,000	25,000	…	25,000
0	1	2	3	…	20

Suppose that selected focal date is the start of the term. Since the focal date is the start of the term, compute for the present value of each offer.

Company A offer:

The present value of P150,000 three years from now is

$$P_1 = F(1 + j)^{-n}$$
$$P_1 = 150000(1+0.04)^{-6}$$
$$P_1 = P118,547.18$$

The present value of P300,000 five years from now is

$$P_2 = F(1 + j)^{-n}$$
$$P_2 = 300000(1+0.04)^{-10}$$
$$P_2 = P202,669.25$$

Fair Market Value(FMV) $= P_1 + P_2$

$= 118547.18 + 202669.25$

FMV $= P321,216.43$

Company B offer: Compute for the present value of a general annuity with quarterly payments but with semi-annual compounding at 8%.

Solve the equivalent rate, compounded quarterly, of 8% compounded semi-annually.

$$F_1 = F_2$$

$$P\left(1 + \frac{i^{(4)}}{4}\right)^{(4)(5)} = P\left(1 + \frac{i^{(2)}}{2}\right)^{(2)(5)}$$

$(1 + \frac{i^{(4)}}{4})^{20} = (1 + \frac{0.08}{2})^{10}$

$1 + \frac{i^{(4)}}{4} = (1.04)^{(1/2)}$

$\frac{i^{(4)}}{4} = (1.04)^{(1/2)} - 1$

$\frac{i^{(4)}}{4} = 0.019803903$

The present value of an annuity is given by

$$P = R \quad \frac{1 - (1 + j)^{-n}}{j}$$

$$P = 25000 \quad \frac{1 - (1 + 0.019803903)^{-20}}{0.019803903}$$

P = P409,560.4726

Therefore, Company B offer is preferable since its market value is larger.

Solved Problems

1. ABC Bank pays interest at the rate of 2% compounded quarterly. How much will Ken have in the bank at the end of 5 years if he deposits P3,000 every month?

Given: R = 3,000

n = mt = (12)(5) = 60 payments

$i^{(4)} = 0.02$

Find: F

Solution.

Convert 2% compounded quarterly to its equivalent interest rate for monthly payment interval.

$(1 + \frac{i^{(12)}}{12})^{(12)} = (1 + \frac{i^{(4)}}{4})^{(4)}$

$(1 + \frac{i^{(12)}}{12})^{12} = (1 + \frac{0.02}{4})^{4}$

$(1 + \frac{i^{(12)}}{12})^{12} = (1.005)^{4}$

$1 + \frac{i^{(12)}}{12} = [(1.005)^{4}]^{(1/12)}$

$\frac{i^{(12)}}{12} = (1.005)^{1/3} - 1 = 0.001664 = j$

Find the future value of an ordinary annuity using the computed equivalent rate

$$F = R \frac{(1+j)^n - 1}{j}$$

$$F = 3{,}000 \ \frac{(1+0.001664)^{60} - 1}{0.001664}$$

Answer: F = P189,126.38

2. A sala set is for sale at P16,000 in cash or on monthly installment of P2,950 for 6 months at 12% compounded semi-annually. Which is lower: the cash price or the present value of the installment term?

Given: Cash price: P 16,000
R = 2,950
$i^{(2)} = 0.12$
m = 12
n = (12)(0.5) = 6 monthly installment payments
Find: present value P

Solution.

Convert 12% compounded semi-annually to its equivalent interest rate for each monthly payment interval.

$$(1 + \frac{i^{(12)}}{12})^{(12)} = P(1 + \frac{i^{(2)}}{2})^2$$

$$(1 + \frac{i^{(12)}}{12})^{(12)} = (1 + \frac{0.12}{2})^2$$

$$\frac{i^{(12)}}{12} = (1.06)^{2(1/12)} - 1$$

$$\frac{i^{(12)}}{12} = (1.06)^{1/6} - 1$$

$$\frac{i^{(12)}}{12} = j = 0.009758$$

Find the present value of an ordinary annuity using the equivalent rate j = 0.0009758.

$$P = R \frac{1 - (1+j)^{-n}}{j}$$

$$P = (2{,}950) \ \frac{1 - (1 + 0.009758)^{-6}}{0.009758}$$

Answer: P = P17,110.88

The cash price is lower than the present value of the installment terms.

3. To accumulate a fund of P500,000 in 3 years, how much should Aling Paring deposit in her account every 3 months if it pays an interest of 5.5% compounded annually.

   Given: F = 500,000
   $i^{(1)} = 0.055$
   m = 4
   n =(4)(3) = 12 deposits

   Find: quarterly deposit R

   Solution.

Convert 5.5% compounded annually to its equivalent interest rate for quarterly deposit.

$$\left(1 + \frac{i^{(4)}}{4}\right)^4 = \left(1 + \frac{i^{(1)}}{1}\right)^{(1)}$$

$$\left(1 + \frac{i^{(4)}}{4}\right)^4 = (1 + 0.055)$$

$$\frac{i^{(4)}}{4} = (1.055)^{1/4} - 1$$

$$\frac{i^{(4)}}{4} = j = 0.013475$$

Find the periodic deposit R of an ordinary annuity using the equivalent rate

j = 0.013475.

$$R = F \,/\, \frac{(1+j)^n - 1}{j}$$

$$R = 500{,}000 \,/\, \frac{(1+0.013475)^{12} - 1}{0.013475}$$

Answer: R = P38,668.16

4. Nadine is the beneficiary of P1,000,000 insurance policy. Instead of taking the money as lump sum, she opted to receive a monthly stipend over a period of 10 years. If the insurance policy pays an interest of 5% compounded annually, what will be her monthly stipend?

   Given: p = 1,000,000
   $i^{(1)} = 0.05$
   m = 1
   n = (10)(12) = 120 monthly stipends

   Find: quarterly deposit R

Solution.

Convert 5% compounded annually to its equivalent interest rate for monthly interval.

$(1 + \frac{i^{(12)}}{12})^{12} = (1 + \frac{i^{(1)}}{1})^{(1)}$

$(1 + \frac{i^{(12)}}{12})^{12} = (1 + 0.05)$

$\frac{i^{(12)}}{12} = (1.05)^{1/12} - 1$

$\frac{i^{(12)}}{12} = j = 0.004074$

Find the monthly stipend R of an ordinary annuity using the equivalent rate j = 0.004074.

$$R = P / \frac{1 - (1 + j)^{-n}}{j}$$

$$PR = (1{,}000{,}000) / \frac{1 - (1 + 0.004074)^{-120}}{0.004074}$$

Answer: R = P10,552.28

5. Kat received two offers for investment. The first one is 150,000 every year for 5 years at 9% compounded annually. The other investment scheme is 12,000 per per month for 5 years with the same interest rate. Which fair market value between these offers is preferable?

Solution: Let the focal point be the end of the term, and so, compute for the future value.

First Offer (Ordinary Annuity)
Given: R = 150,000 $i^{(1)} = 0.09$ m = 1 t = 5 n = 5 payments

$$F = R \frac{(1 + j)^{n} - 1}{j}$$

$$F = 150{,}000 \frac{(1 + 0.09)^{5} - 1}{0.09}$$

Answer: F = 897,706.59

Second Offer: (General Annuity)

Given: R = 12,000 $i^{(1)} = 0.09$ m = 1 n = (5)(12) = 60 monthly payments

Convert 9% compounded annually to its equivalent interest rate for monthly payment interval.

$$(1 + \frac{i^{(12)}}{12})^{(12)} = P(1 + \frac{i^{(1)}}{1})^1$$

$$(1 + \frac{i^{(12)}}{12})^{(12)} = (1 + 0.09)$$

$$\frac{i^{(12)}}{12} = (1.09)^{(1/12)} - 1$$

$$\frac{i^{(12)}}{12} = j = 0.00721$$

Find the future value of this investment.

$$F = R\,\frac{(1 + j)^n - 1}{j}$$

$$F = 12{,}000\,\frac{(1 + 0.00721)^{60} - 1}{0.00721}$$

Answer: F = 896,869.86

The two offers are almost similar but the first offer is preferable.

Lesson 29 Supplementary Exercises

A. Find the future value F of the following general annuities.

1. Monthly payments of 3,000 for 4 years with interest rate of 3% compounded quarterly
2. Quarterly payment of 5,000 for 10 years with interest rate of 2% compounded annually
3. Annual payments of P12, 500 with interest rate of 10.5% compounded semi-annually for 6 years
4. Semi-annual payments of P 105,000 with interest rate of 12% compounded annually for 5 years
5. Daily payments of P 20 for 30 days with interest rate of 20% compounded annually

B. Find the present value P of the following general annuities.

6. Monthly payments of 2,000 for 5 years with interest rate of 12% compounded quarterly
7. Quarterly payment of 15,000 for 10 years with interest rate of 8% compounded annually

8. Annual payments of P20,500 with interest rate of 8.5% compounded semi-annually for 3 years
9. Semi-annual payments of P 150,000 with interest rate of 8% compounded annually for 10 years
10. Daily payments of P 54 for 30 days with interest rate of 15% compounded annually

C. Find the periodic payments of the following general annuities

11. Monthly payment of the future value of P50,000 for 1 year with an interest rate of 10% compounded quarterly
12. Quarterly payment of an accumulated amount of P80,000 for 2 years with interest rate of 8% compounded annually
13. Annual payment for the present value of P100,000 for 2 years with an interest rate of 12% compounded semi-annually
14. Semi-annual payment of the loan P800,000 for 5 years with an interest rate of 9% compounded annually.
15. Monthly installment of an appliance cash prize of P20,000 for 6 months with an interest rate of 6% compounded semi-annually

D. Answer the following problems.

16. Teacher Kaye is saving P2,000 every month by depositing it in a bank that gives an interest of 1% compounded quarterly. How much will she save in 5 years?
17. Vladimir purchased a new car for P99,000 downpayment and P15,000 every month. If the payments are based on 7% compounded quarterly what is the total cash price of his car?
18. In order to have a fund of 1,000,000 at the end of 12 years, equal deposits every six months must be made. Find the semi-annual payment if interest is at 6% compounded annually.
19. Which investment is preferable? (Hint: Compute for the market values)
    Investment in Sunrise company 100,000 at the end of 5 years plus P 24,000 annually for 4 years afterwards. Investment in XYZ company B offers P 50,000 semi-annually 15,000 every 6 months after 6 years . Assume that money is worth 9% compounded annually.
20. A motorcycle is for sale P 60,500 cash or on installment terms 3,000 per month for 2 years at 12% compounded annually. If you were the buyer, what would you prefer, cash or installment?

Lesson 30: Deferred Annuity

Learning Outcome(s): At the end of the lesson, the learner is able to calculate the present value and period of deferral of a deferred annuity

Lesson Outline:

1. Deferred Annuity
2. Present Value of a Deferred Annuity
3. Period of Deferral of a Deferred Annuity

Definition of Terms

Deferred Annuity – an annuity that does not begin until a given time interval has passed

Period of Deferral – time between the purchase of an annuity and the start of the payments for the deferred annuity

Time Diagram for a Deferred Annuity

	R*	R* …	R*	R	R …	R
0	1	2 …	k	k+1	k+2	k+n

In this time diagram the period of deferral is k because the regular payments of R start at time k + 1.

The notation $R*$ represent k "artificial payments," each equal to R, but are not actually paid during the period of deferral.

To determine the present value of a deferred annuity, find the present value of all k + n payments (including the artificial payments), then subtract the present value of all artificial payments.

Present Value of a Deferred Annuity

The present value of a deferred annuity is given by

$$P = R\frac{1-(1+j)^{-(k+n)}}{j} - R\frac{1-(1+j)^{-k}}{j}$$

where
R is the regular payment;
j is the interest rate per period;
n is the number of payments ;
k is the number of conversion periods in the deferral

Example 1. On his 40th birthday, Mr. Ramos decided to buy a pension plan for himself. This plan will allow him to claim P10,000 quarterly for 5 years starting 3 months after his 60th birthday. What one-time payment should he make on his 40th birthday to pay off this pension plan, if the interest rate is 8% compounded quarterly?

Given: R=10,000 m=4 $i^{(4)} = 0.08$
Find: P
Solution.
The annuity is deferred for 20 years and it will go on for 5 years. The first payment is due three months (one quarter) after his 60th birthday, or at the end of the 81st conversion period. Thus, there are 80 artificial payments.

Number of artificial payments: $k = mt = (4)(20) = 80$
Number of actual payments: $n = mt = (4)(5) = 20$

Interest rate per period: $j = \frac{i^{(4)}}{m} = \frac{0.08}{4} = 0.02$

If you assume that there are payments in the period of deferral, there would be a total of $k + n = 80 + 20 = 100$ payments.
Time Diagram:

						10,000	10,000	…	10,000
0	1	2	…	80		81	82	…	100

Thus, the present value of the deferred annuity can be solved as

$$P = R\ \frac{1-(1+j)^{-(k+n)}}{j} - R\ \frac{1-(1+j)^{-k}}{j}$$

$$= 10000\ \frac{1-(1+0.02)^{-100}}{0.02} - 10000\ \frac{1-(1+0.02)^{-80}}{0.02} = 33{,}538.38$$

Therefore, the present value of these monthly pensions is P33,538.38.

Example 2. A credit card company offers a deferred payment option for the purchase of any appliance. Rose plans to buy a smart television set with monthly payments of P4,000 for 2 years. The payments will start at the end of 3 months. How much is the cash price of the TV set if the interest rate is 10% compounded monthly?

Given: R=4,000 m=12 $i^{(12)} = 10$

Find: P

Solution. The annuity is deferred for 2 months and it will go on for 2 years. The first payment is due at the end of 3 months, or at the end of the 3rd conversion period. Thus, there are 2 artificial payments.

Number of artificial payments: $k = 2$

Number of actual payments: $n = mt = (12)(2) = 24$

Interest rate per period: $j = \frac{i^{(12)}}{m} = \frac{0.10}{12} = 0.00833$

If you assume that there are payments in the period of deferral, there would be a total of $k + n = 2 + 24 = 26$ payments.

Time Diagram:

			4000	4000	…	4000
0	1	2	3	4	…	26

Thus, the present value of the deferred annuity can be solved as

$$P = R\ \frac{1-(1+j)^{-(k+n)}}{j} - R\ \frac{1-(1+j)^{-k}}{j}$$

$$= 4000\ \frac{1-(1+0.00833)^{-26}}{0.00833} - 4000\ \frac{1-(1+0.00833)^{-2}}{0.00833} = 85{,}260.53$$

Therefore, the present value of these monthly pensions is P85,260.53.

Solved Examples

In numbers 1 to 5, find the period of deferral in the deferred annuity

1. Monthly payments of P50,000 for 3 years that will start 8 months from now
Solution. The first payment is at time 8. The period of deferral is from time 0 to 7, which is equivalent to 7 periods or 7 months.

2. Annual payments of P2,500 for 24 years that will start 12 years from now

Solution. The first payment is at time 12. The period of deferral is from time 0 to 1, which is equivalent to 11 periods or 11 years.

3. Quarterly payments of 300 for 9 years that will start 1 year from now

Solution. The first payment is at time 4 because there are 4 quarters in 1 year. The period of deferral is from time 0 to 3, which is equivalent to 3 periods or 3 quarters.

4. Semi-annual payments of 6,000 for 13 years that will start 4 years from now

Solution. The first payment is at time 8. The period of deferral is from time 0 to 7, which is equivalent to 7 periods or 7 semi-annual intervals.

5. Payments of 10,000 every 2 years for 30 years starting at the end of 16 years

Solution. The first payment is at time 8 because there is one payment in every two-year period. The period of deferral is from time 0 to 7, which is equivalent to 7 periods or 7 two-year intervals.

6. Melwin availed of a loan from a bank that gave him an option to pay P20,000 monthly for 2 years. The first payment is due after 4 months. How much is the present value of the loan if the interest rate is 10% converted monthly?

Given: R=20,000 m=12 $i^{(12)} = 0.10$ k=3 $n = mt = (12)(2) = 24$

$$j = \frac{i^{(m)}}{m} = \frac{0.10}{12}$$

Find: P

Solution. The present value of the deferred annuity can be solved as

$$P = R\,\frac{1-(1+j)^{-(k+n)}}{j} - R\,\frac{1-(1+j)^{-k}}{j}$$

$$= 20000\,\frac{1-\left(1+\frac{0.10}{12}\right)^{-27}}{\frac{0.10}{12}} - 20000\,\frac{1-\left(1+\frac{0.10}{12}\right)^{-3}}{\frac{0.10}{12}} = 422{,}759.78$$

Therefore, the present value of these payments is P422,759.78.

7. Mariel purchased a smart television set through the credit cooperative of their company. The cooperative provides an option for a deferred payment. Mariel decided to pay after 2 months of purchase. Her monthly payment is computed as P3,800 payable in 12 months. How much is the cash value of the television set if the interest rate is 12% convertible monthly?

Given: R=3,800 m=12 $i^{(12)} = 0.12$ k=1 $n = 12$

$$j = \frac{i^{(m)}}{m} = \frac{0.12}{12} = 0.01$$

Find: P

Solution. The present value of the deferred annuity can be solved as

$$P = R\,\frac{1-(1+j)^{-(k+n)}}{j} - R\,\frac{1-(1+j)^{-k}}{j}$$

$$= 3800\,\frac{1-(1+0.01)^{-13}}{0.01} - 3800\,\frac{1-(1+0.01)^{-1}}{0.01} = 42{,}345.84$$

Therefore, the present value of these payments is P42,345.84 .

8. Mr. Quijano decided to sell their farm and to deposit the fund in a bank. After computing the interest, they learned that they may withdraw P480,000 yearly for 8 years starting at the end of 6 years when it is time for him to retire. How much is the fund deposited if the interest rate is 5% converted annually?

Given: R=480,000 m=1 $j = 0.05$ k=5 $n = 8$

Find: P

Solution. The present value of the deferred annuity can be solved as

$$P = R\,\frac{1-(1+j)^{-(k+n)}}{j} - R\,\frac{1-(1+j)^{-k}}{j}$$

$$= 480000\,\frac{1-(1+0.05)^{-13}}{0.05} - 480000\,\frac{1-(1+0.05)^{-5}}{0.05} = 2{,}430{,}766.23$$

Therefore, the present value of these withdrawals is P2,430,766.23.

9. A group of medical students decided to invest the money they earned from the fund-raising project. After 6 months from today, they want to withdraw from this fund P10,000 quarterly for 1 year to fund their medical mission. How much is the total deposit now if the interest rate is 4% converted quarterly?

Given: R=10,000 m=4 $i^{(4)} = 0.04$ k=1 $n = 4$

$$j = \frac{i^{(m)}}{m} = \frac{0.04}{4} = 0.01$$

Find: P

Solution. The present value of the deferred annuity can be solved as

$$P = R\ \frac{1-(1+j)^{-(k+n)}}{j} - R\ \frac{1-(1+j)^{-k}}{j}$$

$$= 10000\ \frac{1-(1+0.01)^{-5}}{0.01} - 10000\ \frac{1-(1+0.01)^{-1}}{0.01} = 38{,}633.32$$

Therefore, the present value of these payments is P38,633.32.

10. Bella converted her loan to light payments which gives her an option to pay P12,000 every 2 years for 6 years. The first payment is due 4 years from now. How much is the amount of the loan if the interest rate is 10% converted every 2 years?

Given: R=12,000

$m = \frac{1}{2}$ (since payments are made every 2 years, it is equivalent to ½ period in every year)

$i^{(1/2)} = 0.10$ k=1 $n = mt = \left(\frac{1}{2}\right)(6) = 3$ $j = \frac{i^{(m)}}{m} = (0.10)(2) = 0.20$

Find: P

Solution. The present value of the deferred annuity can be solved as

$$P = R\ \frac{1-(1+j)^{-(k+n)}}{j} - R\ \frac{1-(1+j)^{-k}}{j}$$

$$= 12000\ \frac{1-(1+0.20)^{-4}}{0.20} - 12000\ \frac{1-(1+0.20)^{-1}}{0.20} = 21{,}064.81$$

Therefore, the present value of these payments is P21,064.81 .

Lesson 30 Supplementary Exercises

A. Find the period of deferral in each of the following deferred annuity problem.

1. Monthly payments of P1,000 for 9 years that will start 9 months from now
2. Monthly payments of P200 for 6 years that will start at the end of 10 years
3. Semi-annual payments of P12,700 for 5 years that will start 2 years from now
4. Semi-annual payments of P8,500 for 8 years that will start 12 years from now
5. Withdrawals of P7,200 every 3 months for 9 years that will start at the end of 2 years
6. Payments of P13,000 every 3 months for 18 years that will start four years from now

7. Annual payments of P600 for 7 years that will start 7 years from now
8. An amount of P1,850 payable every year for 30 years that will start 5 years from now
9. Payments of P17,000 every 4 years for 12 years starting at the end of 12 years
10. Payments of P20,000 every 3 years for 15 years starting at the end of 12 years

B. Solve the following problems completely.

1. A loan is to be repaid quarterly for 5 years that will start at the end of 2 years. If interest rate is 6% converted quarterly, how much is the loan if the quarterly payment is P10,000?
2. A cash loan is to be repaid by paying P13,500 quarterly for 3 years starting at the end of 4 years. If interest rate is 12% convertible quarterly, how much is the cash loan?
3. A car is to be purchased in monthly payments of P19,500 for 5 years starting at the end of 3 months. How much is the cash value of the car if the interest rate used is 10% converted monthly?
4. A school service van is available for purchase at P23,000 monthly payable in 4 years. The first payment is due in 4 months. How much is the present value of the van if the interest rate applied is 14% converted monthly?
5. A house and lot is to be purchased by paying P35,500 monthly for 25 years. The first payment is due in 1 year. How much is the cash price of the house and lot if the interest rate is 12% converted monthly?
6. A condominium unit is available at P29,000 monthly payable in 20 years. The first payment is due in 2 years exactly the time when the condominium will be turned in to the buyers. How much is the cash value if the interest rate is 10% convertible monthly?
7. A savings account may allow the owner to withdraw P30,000 semi-annually for 3 years starting at the end of 3 years. How much is the savings if the interest rate is 4% converted semi-annually?
8. Mr. Canlapan deposited his money from selling his old vehicle. The fund would allow him to withdraw P45,000 semi-annually for 5 years starting at the end of 1 year. How much is the amount deposited if the interest rate is 2% converted semi-annually?
9. A cellular phone may be purchased at P1,500 payable monthly for 18 months. The first payment is due after 3 months. How much is the cellular phone if the interest rate is 12% convertible monthly?
10. Ruben bought a laptop that is payable by monthly installment of P1,800 for 12 months starting at the end of 2 months. How much is the cash value of the laptop if interest is at 10% convertible monthly?

Lessons 23 – 30 Topic Test 1

1. Mr. Reyes borrowed P50,000 from a lending firm to start a mini-store business. If the firm charges simple interest of 5%, how much must Mr. Reyes pay in three years?
[5]

2. Vlad opens a bank account that gives a simple interest of 2%. How much should Vlad invest if he wants to save P20,000 after 2 years? **[5]**

3. In preparation for Marie's college education, her parents want to save P400,000 after 12 years. How much should they deposit in a bank providing an interest rate of 2% compounded quarterly? **[5]**

4. How long should P50,000 be invested at 6% compounded semi-annually if it should earn an interest of P10,000? **[5]**

5. Kat is choosing between two short term investments. She can invest at 10% simple interest per year for 1½ years or 9.5% compounded monthly for the same term. Which investment should she choose? Why? **[10]**

6. Shirl deposits P10,000 every 3 months in a time deposit account giving 2% interest rate compounded quarterly. How much will she save at the end of 5 years? How much interest is earned in all deposits? **[10]**

7. Mr. Bautista bought a car and gave an initial payment of P180,000 as down payment. The remaining balance is to be settled by paying P18,000 at the end of each month for 5 years. If interest is 10% compounded monthly, what is the cash price of his car?
[10]

8. A teacher will be retiring in 15 years. At her retirement, she wants to save a fund of one million pesos. He invested at a fund that gives 4% compounded semi-annually. How much should she deposit every 6 months in order to have this amount upon retirement?
[10]

9. Mr. Cama borrowed one million pesos from a bank to buy a house which he would pay in 15 years. How much is the monthly payment for this loan if 5% interest compounded annually is charged against the loan? **[15]**

10. Reggie availed of a deferred payment scheme from a bank that gave her an option to pay P5,500 monthly for 2 years. The first payment is due after 3 months. How much is the present value of the loan if the interest rate is 12% converted monthly? **[15]**

Lessons 23 – 30 Topic Test 2

1. Blad deposited P8,000 in a bank that gives 2.5% simple interest per year. How long will he wait if he wants it to accumulate to P10,000? **[5]**

2. At what simple interest rate much P100,000 be invested so as to gain an interest of P10,000 after 5 years? **[5]**

3. James borrowed P16,000 from Nadine at an interest rate of 8% compounded semi-annually. How much does he need to pay after 3 years? **[5]**

4. At what interest rate compounded quarterly will a certain amount double in 10 years? **[5]**

5. Angel has P100,000 which she plans to invest in 4 years. She is choosing between two offers. Investment A gives 6.4% compounded semi-annually while investment B provides 6% compounded monthly. Which investment should she choose? Why? **[10]**

6. A teacher will be retiring in 15 years. At her retirement, she wants to save a fund of one million pesos. He invested at a fund that gives 4% compounded semi-annually. How much should she deposit every 6 months in order to have this amount upon retirement? **[10]**

7. Cris is planning to buy a house and lot. The required down payment is P300,000. His balance will be paid at P15,200 every month for 20 years. If money is 12% compounded monthly, how much is the cash value of the house and lot? **[10]**

8. Christian wants to buy a new computer worth P35,000. He opted to pay it in equal monthly payments for 3 months at 6 % compounded monthly. How much is his monthly payment? How much is the total interest in all payments**.** **[10]**

9. Mrs. Paro invests P5,000 every 3 months at an interest rate of 6 % compounded annually. How much will she have in this investment at the end of 6 years? How much interest is earned? **[10]**

10. Roy purchased a house and lot by paying a down payment of P400,000 and P20,000 monthly for 25 years. The first payment is due after one year. How much is the cash price of the house and lot if the interest rate is 10% converted monthly? **[10]**

Lesson 31: Stocks and Bonds

Learning Outcome(s): At the end of the lesson, the learner is able to:

1. Illustrate Stocks and Bonds
2. Distinguish between Bonds and Stocks

Lesson Outline:

1. Definition of terms related to stocks
2. Definition of terms related to bonds

STOCKS[12]

Some corporations may raise money for their expansion by issuing stocks. Stocks are shares in the ownership of the company. Owners of stocks may be considered as part owners of the company. There are two types of stocks: common stock and preferred stock. Both will receive dividends or share of earnings of the company. Dividends are paid first to preferred shareholders.

Stocks can be bought or sold at its current price called the market value. When a person buys some shares, the person receives a certificate with the corporation's name, owner's name, number of shares and par value per share.

BONDS[13]

Bonds are interest bearing security which promises to pay amount of money on a certain maturity date as stated in the bond certificate. Unlike the stockholders, bondholders are lenders to the institution which may be a government or private company. Some bond issuers are the national government, government agencies, government owned and controlled corporations, non-bank corporations, banks and multilateral agencies. Bondholders do not vote in the institution's annual meeting but the first to claim in the institution's earnings. On the maturity date, the bondholders will receive the face amount of the bond. Aside from the face amount due on the maturity date, the bondholders may receive coupons (payments/interests), usually done semi-annually, depending on the coupon rate stated in the bond certificate.

[12] Stock Basics Tutorial, Accessed from http://www.investopedia.com/university/stocks/
[13] Bonds Basics Tutorial, Accessed from http://www.investopedia.com/university/bonds/

Comparison of Stocks and Bonds[14]

Stocks	Bonds
A form of **equity financing** or raising money by allowing investors to be part owners of the company.	A form of **debt financing**, or raising money by borrowing from investors
Stock prices vary every day. These prices are reported in various media (newspaper, TV, internet, etc).	Investors are guaranteed interest payments and a return of their money at the maturity date
Investors can earn if the stock prices increase, but they can lose money if the stock prices decrease or worse, if the company goes bankrupt.	Investors still need to consider the borrower's credit rating. Bonds issued by the government pose less risk than those by companies because the government has guaranteed funding (taxes) from which it can pay its loans.
Higher risk but with possibility of higher returns	Lower risk but lower yield
Can be appropriate if the investment is for the long term (10 years or more). This can allow investors to wait for stock prices to increase if ever they go low.	Can be appropriate for retirees (because of the guaranteed fixed income) or for those who need the money soon (because they cannot afford to take a chance at the stock market)

[14] Investopedia staff (n.d.) *Bond basics: What are bonds?* (http://www.investopedia.com/university/bonds/bonds1.asp)

Definition of Terms in Relation to Stocks

Stocks –share in the ownership of a company

Dividend – share in the company's profit

Dividend Per Share –ratio of the dividends to the number of shares

Stock Market –a place where stocks can be bought or sold. The stock market in the Philippines is governed by the Philippine Stock Exchange (PSE)

Market Value –the current price of a stock at which it can be sold

Stock Yield Ratio –ratio of the annual dividend per share and the market value per share. Also called current stock yield.

Par Value –the per share amount as stated on the company certificate. Unlike market value, it is determined by the company and remains stable over time

Example 1. A certain financial institution declared a P30,000,000 dividend for the common stocks. If there are a total of 700,000 shares of common stock, how much is the dividend per share?

Given: Total Dividend = P30,000,000

Total Shares = 700,000

Find: Dividend per Share

Solution.

$$\text{Dividend per Share} = \frac{\text{Total Dividend}}{\text{Total Shares}}$$

$$= \frac{30{,}000{,}000}{700{,}000}$$

$$= 42.86$$

Therefore, the dividend per share is P42.86.

Example 2. A certain corporation declared a 3% dividend on a stock with a par value of P500. Mrs Lingan owns 200 shares of stock with a par value of P500. How much is the dividend she received?

Given: Dividend Percentage = 3%

Par Value = P500

Number of Shares = 200

Find: Dividend

Solution.

The dividend per share is: P500 x 0.03 = P15.

Since there are 300 shares, the total dividend is:

P15/share x 200 shares = P3,000

In summary,

Dividend=(Dividend Percentage)x(Par Value) x(No. of Shares)

= (0.03)(500)(200)

= 3,000

Thus, the dividend is P3,000.

Example 3. Corporation A, with a current market value of P52, gave a dividend of P8 per share for its common stock. Corporation B, with a current market value of P95, gave a dividend of P12 per share. Use the **stock yield ratio** to measure how much dividends shareholders are getting in relation to the amount invested.

Solution.

Given: <u>Corporation A</u>:

Dividend per share = P8

Market value = P52

Find: stock yield ratio

Stock yield ratio $= \frac{\text{dividend per share}}{\text{market value}}$

$$= \frac{8}{52}$$

$$= 0.1538 = 15.38\%$$

Corporation B:

Dividend per share = P12

Market value = P95

Find: stock yield ratio

$$\text{Stock yield ratio} = \frac{\text{dividend per share}}{\text{market value}}$$

$$= \frac{12}{95}$$

$$= 0.1263 = 12.63\%$$

Corporation A has a higher stock-yield-ratio than Corporation B. Thus, each peso would earn you more if you invest in Corporation A than in Corporation B. If all other things are equal, then it is wiser to invest in Corporation A.

As Example 3 shows, the stock yield ratio can be used to compare two or more investments.

Definition of Terms in Relation to Bonds

Bond – interest-bearing security which promises to pay (1) a stated amount of money on the maturity date, and (2) regular interest payments called coupons.

Coupon –periodic interest payment that the bondholder receives during the time between purchase date and maturity date; usually received semi-annually

Coupon Rate –the rate per coupon payment period; denoted by r

Price of a Bond –the price of the bond at purchase time; denoted by P

Par Value or Face Value - the amount payable on the maturity date; denoted by F.

If P = F, the bond is **purchased at par**.

If P < F, the bond is **purchased at a discount**.

If P > F, the bond is **purchased at premium**.

Term of a Bond – fixed period of time (in years) at which the bond is redeemable as stated in the bond certificate; number of years from time of purchase to maturity date.

Fair Price of a Bond –present value of all cash inflows to the bondholder.

Example 4. Determine the amount of the semi-annual coupon for a bond with a face value of P300,000 that pays 10%, payable semi-annually for its coupons.

Given: Face Value F = 300,000

Coupon rate r = 10%

Find: Amount of the semi-annual coupon

Solution.

Annual coupon amount: 300,000(0.10) = 30,000.

Semi-annual coupon amount: $30{,}000\left(\frac{1}{2}\right)=15{,}000$.

Thus, the amount of the semi-annual coupon is P15,000.

> Note: The **coupon rate** is used only for computing the coupon amount, usually paid semi-annually. It is *not* the rate at which money grows. Instead current market conditions are reflected by the **market rate**, and is used to compute the present value of future payments.

Example 5. Suppose that a bond has a face value of P100,000 and its maturity date is 10 years from now. The coupon rate is 5% payable semi-annually. Find the fair price of this bond, assuming that the annual market rate is 4%.

Given: Coupon rate r = 5%, payable semi-annually

Face Value = 100,000

Time to maturity = 10 years

Number of periods = 2(10) = 20

Market rate = 4%

Solution: Amount of semi-annual coupon: $100{,}000\left(\frac{0.05}{2}\right)=2500$

The bondholder receives 20 payments of P2,500 each, and P100,000 at t = 10.

Present value of P100,000:

$$P = \frac{F}{(1+j)^n} = \frac{100000}{(1+.04)^{10}} = 67{,}556.42$$

Present value of 20 payments of P2,500 each:

Convert 4% to equivalent semi-annual rate:

$$(1+0.04)^1 = \left(1+\frac{i^{(2)}}{2}\right)^2$$

$$\frac{i^{(2)}}{2} = 0.019804$$

$$P = R\ \frac{1-(1+j)^{-n}}{j} = 2500\ \frac{1-(1+0.019804)^{-20}}{0.019804} = 40{,}956.01$$

Price = 67,556.42 + 40,956.01 = 108,512.43.

Thus, a price of P108,512.14 is equivalent to all future payments, assuming an annual market rate of 4%.

Solved Examples

1. A financial institution declared a dividend of P75,000,000 for its common stock. Suppose there are 900,000 shares of common stock, how much is the dividend per share?

Solution:

Given: Total Dividend = P75,000,000
Total Shares = 900,000

Find: Dividend per Share

$$\text{Dividend per Share} = \frac{\text{Total Dividend}}{\text{Total Shares}}$$

$$= \frac{75{,}000{,}000}{900{,}000}$$

$$= 83.33$$

Therefore, the dividend per share is P83.33.

2. The ABC corporation gave out P38 dividend per share for its common stock. The market value of the stock is P108. Determine the stock yield ratio.

Solution:

Given: Dividend per share = P38
Market value = P108

Find: stock yield ratio

$$\text{Stock yield ratio } = \frac{\text{dividend}}{\text{market value}}$$

$$= \frac{38}{108}$$

= 0.35

Therefore, the stock yield ratio is 0.35 .

3. A bank declared a dividend of P27 per share for the common stock. If the common stock closes at P93, how large is the stock yield ratio on this investment?

Solution:

Given: Dividend per share = P27

Market value = P93

Find: stock yield ratio

$$\text{Stock yield ratio } = \frac{\text{dividend}}{\text{market value}}$$

$$= \frac{27}{93}$$

= 0.29

Therefore, the stock yield ratio is 0.29 .

4. Find the amount of the semi-annual coupon for a P200,000 bond which pays 5% convertible semi-annually for its coupons .

Solution:

Given: Face Value F = P200,000

Nominal yield i = 5%

Find: Amount of the semi-annual coupon

$$Fr = 200{,}000\left(\frac{0.05}{2}\right) = 5000.$$

Thus, the amount of the semi-annual coupon is P5000.

5. Determine the amount of semi-annual coupon paid for a 3% bond with a face value of P100,000 which matures after 8 years. How many coupons are paid?

Solution:

Given: vvFace Value F = P100,000

Nominal yield *i* = 3%

Find: Number and Amount of each semi-annual coupon

$$Fr = 100{,}000\left(\frac{0.03}{2}\right) = 1500.$$

In 8 years, there are 8 x 2 = 16 payments.

Thus, each semi-annual coupon is P1,500, paid 16 times every six months.

6. A certain bond pays coupons of P5,000 every six months for .

Given: Semi-annual coupons = 1,000

Face Value = 120,000

Time to maturity = 8 years

Number of periods = 2(8) = 16

Market rate = 6%

Solution:

The bondholder receives 16 payments of P5,000 each, and P120,000 at t = 8.

Convert 6% to equivalent annual rate:

$$\left(1+\frac{.06}{3}\right)^2 = \left(1+i^{(1)}\right)$$

$$i^{(1)} = 0.0609$$

Present value of P120,000:

$$P = \frac{F}{(1+j)^n} = \frac{120000}{(1+.0609)^8} = 74{,}780.03$$

Present value of 16 semi-annual payments of P5,000 each with 3% interest per period:

$$P = R\ \frac{1-(1+j)^{-n}}{j} = 5000\ \frac{1-(1+0.03)^{-16}}{0.03} = 62805.51$$

Price = 74,780.03 + 62805.51 = 137,585.54.

Thus, the fair price of the bond isP137,585.54.

Lesson 31 Supplementary Exercises

1. Stockholder A got 4500 shares of stocks from XYZ Corporation. The par value is P 150. How much is the dividend if the percentage is 3%?
2. Mr. Roman purchased 1000 shares of stocks at P25 par value. How much is his dividend if the percentage declared by the company is 2%?
3. A certain land developer declared a dividend of P 23,000,000 for its common stock. Suppose there are 500,000 shares of common stock, how much is the dividend per share?
4. A financial institution will give out a dividend of P57,000,000 for its common stock. Suppose there are 700,000 shares of common stock, how much is the dividend per share?
5. A resort company gave out P11 dividend per share for its common stock. The market value of the stock is P17. Determine the stock yield ratio.
6. A telecommunication company gave out P800 dividend per share for its common stock. The market value of the stock is 1180. Determine the stock yield ratio.
7. Find the amount of the semi-annual coupon for a P110,000 bond which pays 4.5% convertible semi-annually for its coupons .
8. Find the amount of the semi-annual coupon for a P25,000 bond which pays 2.5% convertible semi-annually for its coupons .
9. A bond promises to pay the bondholder equal payments of P6,000 in six-month intervals for 30 years. If the face amount is P450,000, what is the fair price of the bond? Assume that the market rate is 2% compounded annually.

Lesson 32: Market Indices for Stocks and Bonds

Learning Outcome(s): At the end of the lesson, the learner is able to describe the different markets for stocks and bonds, and analyze the different market indices for stocks and bonds

Lesson Outline:

1. Market Indices for Stocks
2. Market Indices for Bonds

Definition: A **stock market index** is a measure of a portion of the stock market.

One example is the PSE Composite Index or PSEi. It is composed of 30 companies[15] carefully selected to represent the general movement of market prices.

[15] http://www.pse.com.ph/stockMarket/home.html

The up or down movement in percent change over time can indicate how the index is performing.

Other indices are **sector indices**, each representing a particular sector (e.g., financial institutions, industrial corporations, holding firms, service corporations, mining/oil, property)[16].

The stock index can be a standard by which investors can compare the performance of their stocks. A financial institution may want to compare its performance with those of others. This can be done by comparing with the "financials" index.

Stock Index Tables

Stock indices are reported in the business section of magazines or newspapers, as well as online (http://www.pse.com.ph/stockMarket/home.html). The following table shows how a list of index values is typically presented (values are hypothetical).

Index	**Val**	**Chg**	**%Chg**
PSEi	7,523.93	–14.20	–0.19
Financials	4,037.83	6.58	0.16
Holding Firms	6,513.37	2.42	0.037
Industrial	11,741.55	125.08	1.07
Property	2,973.52	–9.85	–0.33
Services	1,622.64	–16.27	–1.00
Mining and Oil	11,914.73	28.91	0.24

Val – value of the index

Chg – change of the index value from the previous trading day (i.e., value today minus value yesterday)

%Chg – ratio of Chg to Val (i.e., Chg divided by Val)

Stock Tables

Various information about stock prices can be reported. The following table shows how information about stocks can be presented (values are hypothetical).

52-WK HI	52-WK LOW	STOCK	HI	LO	DIV	VOL(100s)	CLOSE	NETCHG
94	44	AAA	60	35.5	.70	2050	57.29	0.10
88	25	BBB	45	32.7	.28	10700	45.70	–0.2

[16] Ibid.

52-WK HI/LO – highest/ lowest selling price of the stock in the past 52 weeks

HI/LO – highest/ lowest selling price of the stock in the last trading day

STOCK – three-letter symbol the company is using for trading

DIV – dividend per share last year

VOL (100s) – number of shares (in hundreds) traded in the last trading day. In this case, stock AAA sold 2,050 shares of 100 which is equal to 20,500 shares.

CLOSE- closing price on the last trading day.

NETCHG- net change between the two last trading days. In the case of AAA, the net change is 0.10. The closing price the day before the last trading day is
P57.29 – P0.10 = P57.19.

Buying or Selling Stocks

To buy or sell stocks, one may go to the PSE personally. However, most transactions nowadays are done by making a phone call to a registered broker or by logging on to a reputable online trading platform. Those with accounts in online trading platforms may often encounter a table such as the following.

Bid			**Ask/Offer**		
Size		Price	Price	Size	
122	354,100	21.6000	21.8000	20,000	1
9	81,700	21.5500	21.9000	183,500	4
42	456,500	21.5000	22.1500	5,100	1
2	12,500	21.4500	22.2500	11,800	4
9	14,200	21.4000	22.3000	23,400	6

In the table, the terms mean the following:

- **Bid Size** – the number of individual buy orders and the total number of shares they wish to buy
- **Bid Price** – the price these buyers are willing to pay for the stock
- **Ask Price** – the price the sellers of the stock are willing to sell the stock
- **Ask Size** – how many individual sell orders have been placed in the online platform and the total number of shares these sellers wish to sell.

For example, the first row under Bid means that there are a total of 122 traders who wish to buy a total of 354,000 shares at P21.60 per share. On the other hand, the first row under Ask means that just one trader is willing to sell his/her 20,000 shares at a price of P21.80 per share.

Bond Market Indices

Definition: A **bond market index** is a measure of a portion of the bond market.

The main platform for bonds or fixed income securities in the Philippines is the Philippine Dealing and Exchange Corporation (or PDEx). Unlike stock indices which are associated with virtually every stock market in the world, bond market indices are far less common. In fact, other than certain regional bond indices which have sub-indices covering the Philippines, our bond market does not typically compute a bond market index. Instead, the market rates produced from the bond market are interest rates which may be used as benchmarks for other financial instruments.

The Bond Market and Government Bonds

Government bonds are auctioned out to banks and other brokers and dealers every Monday by the Bureau of Treasury. Depending on their terms (or tenors), these bonds are also called treasury bills (t-bills), treasury notes (t-notes), or treasury bonds (t-bonds). The resulting coupon rates and the total amount sold for these bonds are usually reported by news agencies on the day right after the auction. Since these bond transactions involve large amounts, these bonds are usually limited to banks, insurance firms, and other financial institutions. The banks may then re-sell these bonds to its clients.

Although the coupon rate for bonds is fixed, bond prices fluctuate because they are traded among investors in what is called the secondary market. These prices are determined by supply and demand, the prevailing interest rates, as well as other market forces. As the price of the bond may increase or decrease, some investors may choose to sell back to banks the bonds they acquired before their maturity to cash in their gains even before maturity.

Despite the fact that bond investing is considered safer than stock investing, there is still some risk involved. The most extreme scenario is default by the issuer. In this case, the investor can lose not only the coupons, but even the money invested in the bond. Bond investors should thus be aware of the financial condition of the issuer of the bond and of prevailing market conditions.

Solved Example

1. Consider the following listing on stocks and answer the questions that follow:

52 weeks							
HI	LO	STOCK	DIV	YLD%	VOL(100s)	CLOSE	NETCHG
120	105	GGG	3.5	2.8	4050	118.50	-0.50
16	12	HHH	0.9	1.1	1070	15.80	0.10

For Stocks GGG and HHH:

1. What was the lowest price of the stock for the last 52 weeks?
2. What was the dividend per share last year?
3. What was the annual percentage yield last year?
4. What was the closing price in the last trading day?
5. What was the closing price the day before the last trading day?

Answers:
For Stock GGG:
1. Lowest Price = P 105.00
2. Dividend per Share = P3.50
3. YLD% = 2.8%
4. Closing Price = P 118.50
5. Closing Price (the day before the last trading day) = P 118.50 +P 0.50
   = P 119.00

For Stock HHH:
1. Lowest Price = P 12.00
2. Dividend per Share = P0.90
3. YLD% = 1.1%
4. Closing Price = P15.80
5. Closing Price (the day before the last trading day) = P15.80 - P 0.10
   = P 15.70

Lesson 32 Supplementary Exercises

Consider the following listing on stocks and answer the questions that follow:

52 weeks							
HI	LO	STOCK	DIV	YLD%	VOL(100s)	CLOSE	NETCHG
75	65	JJJ	2.5	2.8	1500	70	2
34	23	KKK	1.7	1.75	1200	28	-3

For Stock JJJ :

1. What was the highest price of the stock for the last 52 weeks?
2. What was the dividend per share last year?
3. What was the annual percentage yield last year?
4. What was the closing price in the last trading day?
5. What was the closing price the day before the last trading day?

For Stock KKK:

6. What was the lowest price of the stock for the last 52 weeks?
7. How many shares were traded in the last trading day?
8. What was the dividend per share?
9. What was the closing price in the last trading day?
10. What was the closing price the day before the last trading day?

Lesson 33: Theory of Efficient Markets

Learning Outcome(s): At the end of the lesson, the learner is able to interpret the theory of efficient markets.

Lesson Outline:

1. The Efficient Market Hypothesis
2. Different Types of Efficient Markets

Definition of Terms

Fundamental Analysis–analysis of various public information (e.g., sales, profits) about a stock

Technical Analysis –analysis of patterns in historical prices of a stock

Weak Form of Efficient Market Theory –asserts that stock prices already incorporate all past market trading data and information (historical price information) *only*

Semistrong Form of Efficient Market Theory –asserts that stock prices already incorporate all publicly available information *only*

Strong Form of Efficient Market Theory –asserts that stock prices already incorporate all information (public and private)

The Efficient Market Hypothesis

The theory of efficient markets was developed by Eugene Fama in the 1970's. It says that stock prices already reflect all the available information about the stock.[17] This means that stock prices are "accurate"—they already give a correct measure of the value of a stock precisely because the prices are already based on all information and expectation about the stock.

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[17] http://www.investopedia.com/terms/e/efficientmarkethypothesis.asp

The slogan **“Trust market prices!”** can sum up the theory. One can trust market prices because they give an accurate measure of all possible information about the stock.

Since all stocks are “correctly priced” (because they are based on all available information), then there is no such thing as discovering undervalued or overvalued stocks from which to gain profits. Thus, the theory implies that **investors cannot beat the market even if they do a lot of research**. In the end, investors will just find out that the correct price is what is already published.

There are three form of efficient market, as discussed by Clarke, Jandik, and Mandelker[18].

For the **weak form of the theory**, stock prices already reflect all past market trading data and historical information *only*. Thus, knowing past data will not give investors an edge. If the weak form of the theory is true, then a **technical analysis** (an analysis of past prices) will not yield new information and hence will not lead to systematic profits.

For **semistrong form of the theory**, stock prices already reflect all publicly available data, including those involving the product, management team, financial statement, competitors and industry. If the semistrong form of the theory is true, then doing a **fundamental analysis** (gathering all public data) will still not lead to systematic profits.

For the **strong form of the theory**, all information (public and private) are incorporated in the price. If the strong form of the theory is true, then investors still cannot gain systematic profits even if they gather information that is *not* yet publicly known.

Solved Examples

Provide a *counter-argument* to the following statement.

1. We cannot beat the stock market because stock prices already reflect all the given information about the stocks.

Sample counter-argument: Information about stocks can change quickly, and it takes time (and high-speed computers) before a stock price can reflect all information.

2.We can beat the stock market because several people have already gained millions (or even billions) from stock trading.

[18] Clarke, J., Jandik, T., & Mandelker, G. (n.d.) *The efficient markets hypothesis.* http://m.e-m-h.org/ClJM.pdf

Sample counter-argument: These people could just be lucky. By the theory of efficient markets, investors cannot systematically gain from the stock market even if they do a lot of research.

3. One can beat the stock market because stock prices fluctuate very often (every day, hour, and minute), and they can be overvalued or undervalued.

Sample counter-argument: The theory of efficient markets states that all information is incorporated right away and constantly. Thus stock prices tend to respond quickly.

4. One can beat the stock market by gathering more information about stocks to determine the best place to invest.

Sample counter-argument: The theory of efficient market states that all the needed information, public or private, are already incorporated in stock price.

5. One can beat the stock market by obtaining the services of financial analysts.

Sample counter-argument: Financial analysts may help in the analysis of stock prices especially on researching on mispriced stocks. But financial analysis may be costly. Some say that the gain may not be enough to pay the cost of a financial analysis.

Lesson 33 Supplementary Exercises

True or False.

1. Stocks are shares in the ownership of a company.
2. The theory of efficient markets states that prices of investments reflect all available information.
3. The weak form of the theory of efficient markets states that only all public information are incorporated in the price of stocks.
4. The semistrong form of the theory of efficient markets states that only all past available information are incorporated in the price.
5. The strong form of the theory of efficient markets states that all information (public and private) are incorporated in the price.
6. Fundamental analysis is the analysis of historical prices.
7. Technical analysis is the analysis of past prices.
8. The theory of efficient market is developed by Eugene Fama.

Lesson 34: Business and Consumer Loans

Learning Outcome(s): At the end of the lesson, the learner is able to illustrate business and consumer loans, and distinguish between business and consumer loans.

Lesson Outline:

1. Business and Consumer Loans

Definition of Terms

Business Loan – money lent specifically for a business purpose. It may be used to start a business or to have a business expansion

Consumer Loan –money lent to an individual for personal or family purpose

Collateral – assets used to secure the loan. It may be real-estate or other investments

Term of the Loan – time to pay the entire loan

In Examples 1-5, identify whether the following is a consumer or business loan.

Example 1. Mr. Agustin plans to have a barbershop. He wants to borrow some money from the bank in order for him to buy the equipment and furniture for the barbershop.

Solution. Business loan

Example 2. Mr and Mrs Craig wants to borrow money from the bank to finance the college education of their son.

Solution. Consumer loan

Example 3. Mr. Alonzo wants to have some improvements on their 10-year old house. He wants to build a new room for their 13-year old daughter. He will borrow some money from the bank to finance this plan.

Solution. Consumer loan

Example 4. Mr. Samson owns a siomai food cart business. He wants to put another food cart on a new mall in the other city. He decided to have a loan to establish the new business.

Solution. Business loan

Example 5. Roan has a computer shop. She owns 6 computers. She decided to borrow some money from the bank to buy 10 more computers.

Solution. Business loan

Lesson 34 Supplementary Exercises

Identify the following whether the following illustrates a business loan or a consumer loan.

1. Mr. Lim wants to have another branch for his cellphone repair shop. He decided to apply for a loan that he can use to pay for the rentals of the new branch.
2. Mr. Trillas runs a trucking business. He wants to buy three more trucks for expansion of his business. He applied for a loan in a bank.
3. Mrs. Alonzo decided to take her family for a vacation. To cover the expenses, she decided to apply for a loan.
4. Glenn decided to purchase a condominium unit near his workplace. He got a loan worth P2,000,000.
5. Mr. Galang renovated her house forP80,000. This was made possible because of an approved loan worth P75,000.

Lesson 35: Solving Problems on Business and Consumer Loans (Amortization and Mortgage)

Learning Outcome(s): At the end of the lesson, the learner is able to solve problems on business and consumer loans (amortization and mortgage).
Lesson Outline:

1. Definition of terms
2. Loan Repayment
3. Interest Amount
4. Mortgage and Amortization
5. Outstanding Balance

Definition of Terms

Amortization Method – method of paying a loan (principal and interest) on installment basis, usually of equal amounts at regular intervals

Mortgage – a loan, secured by a collateral, that the borrower is obliged to pay at specified terms.

Chattel Mortgage – a mortgage on a movable property

Collateral – assets used to secure the loan. It may be a real-estate or other investments

Outstanding Balance – any remaining debt at a specified time

Example 1. Mr. Garcia borrowed P1,000,000 for the expansion of his business. The effective rate of interest is 7%. The loan is to be repaid in full after one year. How much is to be paid after one year?

Solution.

Given: P = 1,000,000 $j = 0.07$

n = 1

Find F.

Solution. $F = P(1+j)^n = 1{,}000{,}000(1+0.07) = 1{,}070{,}000$

An amount of P1,070,000 must be paid after one year.

Example 2 (Chattel mortgage). A person borrowed P1,200,000 for the purchase of a car. If his monthly payment is P31,000 on a 5-year mortgage, find the total amount of interest.

Given: P = 1,200,000 Monthly payment = 31,000

Solution.

The total amount paid is given by

Total Amount = (31,000)(12 months)(5 years)

= 1,860,000

Thus, the total interest is the difference between the total amount paid and the amount of the mortgage;

Total Interest = 1,860,000 – 1,200,000

= 660,000

Example 3. If a house is sold for P3,000,000 and the bank requires 20% down payment, find the amount of the mortgage.

Solution.

Down payment = down payment rate x cash price

= 0.20(3,000,000)

= 600,000

Amount of the Loan = cash price - down payment

= 3,000,000 – 600,000

= 2,400,000

The mortgage amount is P2,400,000.

Example 4. Ms Rosal bought a car. After paying the downpayment, the amount of the loan is P400,000 with an interest rate of 9% compounded monthly. The term of the loan is 3 years. How much is the monthly payment?

Solution.

Given: $P = 400{,}000$ $i^{(12)} = 0.09$, $j = \frac{i^{(12)}}{12} = \frac{0.09}{12} = 0.0075$ n=36

Find: the regular payment R

$$R = \frac{P}{\left[\frac{1-(1+j)^{-n}}{n}\right]} = \frac{400{,}000}{\left[\frac{1-(1+0.0075)^{-36}}{0.0075}\right]} = 12{,}719.89$$

The regular payment is P12,719.89.

Outstanding Balance

Recall that the outstanding balance of a loan is the amount of the loan at this time.

> Note: In this case, the house itself is used as the mortgaged property. Also please take note that the other way to solve this is to directly compute the mortgaged amount by multiplying the cash value of the property by the percentage of the financed amount, which in this case, 100%-20%=80%. Thus, the amount of the loan is given by
>
> (0.80)(P3,000,000)= P2,400,000.

One method to compute the outstanding balance is to get the present value of all remaining payments. This method is called the **prospective method**.

We use the symbol B_k to denote the outstanding balance after k payments. In other books, ${}_pOB_k$ is used (the "p" stands for "prospective").

Example 5. Mrs. Se borrowed some money from a bank that offers an interest rate of 12% compounded monthly. His monthly amortization for 5 years is P11,122.22. How much is the outstanding balance after the 12th payment?

Given: R = 11,122.22 $i^{(12)} = 0.12$ $j = \frac{i^{(12)}}{12} = \frac{0.12}{12} = 0.01$

k=12 number of payments paid

n – k=48since only 48 payments remain

Find: present value of 48 future payments (since there are 48 payments left)

Solution.

$$B_k = R\left[\frac{1-(1+j)^{-(n-k)}}{j}\right] = 11{,}122.22\left[\frac{1-(1.01)^{-48}}{0.01}\right] = 422{,}354.73$$

The outstanding balance is P422,354.73.

Example 6. Mr. and Mrs. Banal purchased a house and lot worth P4,000,000. They paid a down payment of P800,000. They plan to amortize the loan of P3,200,000 by paying monthly for 20 years. The interest rate is 12% convertible monthly.

(a) How much is the monthly payment?
(b) What is the total interest paid?
(c) What are the principal and interest components of the 51st payment?

Solution.

(a) Given: P =3,200,000

$i^{(12)} = 0.12$, $j = \frac{i^{(12)}}{12} = \frac{0.12}{12} = 0.01$, n = mt = (12)(20) = 240

Find: regular payment R

Using the formula $P = R\left[\frac{1-(1+j)^{-n}}{j}\right]$,

then $R = \frac{P}{\left(\frac{1-(1+j)^{-n}}{j}\right)} = \frac{3{,}200{,}000}{\left(\frac{1-(1+0.01)^{-240}}{0.01}\right)} = 35{,}234.76$

Therefore, the monthly payment is P 35,234.76.

(b) Given: P =3,200,000 R = 35,234.76 n = 240

Find: total interest paid

There are 240 payments of P35,234.76. The total payment is

240 × P 35,234.76 = P8,456,342.40.

The principal is only P 3,200,000.

$$\text{Interest Amount} = \text{Total Payments - Principal}$$
$$= 8{,}456{,}342.40 - 3{,}200{,}000 = 5{,}256{,}342.40$$

The interest amount is P5,256,342.40

Note: You may be surprised to learn that much of what is being paid is for the interest. This is particularly true if a loan is being paid over a long period of time.

c) Given: P =3,200,000 $i^{(12)} = 0.12$ $j = \frac{i^{(12)}}{12} = \frac{0.12}{12} = 0.01$

n = mt = (12)(20) = 240

R = 35,234.76

Find: principal and interest components of the 51st payment

The 51st payment of P35,234.76 is partly used to pay for the principal, and partly to pay for the interest.

Step 1: Get the outstanding balance after the 50th payment (the balance after the 50th payment is what the 51st payment will be for).

Since 50 payments have been paid already, there will be 190 remaining payments.

The outstanding balance after the 50th payment is:

$$B_{50} = R\left[\frac{1-(1+j)^{-190}}{j}\right] = 35{,}234.76\left[\frac{1-(1+0.01)^{-190}}{0.01}\right] = 2{,}991{,}477.63$$

Step 2: After the 50th payment, the outstanding balance is P2,991,477.63.

Since the interest rate per period is j = 0.01, then the remaining balance of P2,991,477.63 will be charged an interest of $(0.01)(2{,}991{,}477.63) = 29{,}914.78$

The 51st payment of P35,234.76 will be used to pay for this interest. Thus, the interest component I_{51} of the 51st payment is P29,914.78.

The remaining portion of the 51st payment is the principal component, denoted by PR_{51}, is:

$$PR_{51} = R - I_{50} = 35{,}234.76 - 29{,}914.78 = 5{,}319.98$$

Thus, for the 51st payment, the part that goes to pay the interest is P29,914.78 and the part that goes to pay the principal is P5,319.98.

Solved Examples

1. A loan of P300,000 is to be repaid in full after 2years. If the interest rate is 9% per annum. How much should be paid after 2 years?

Solution.

Given: P = P300,000 $j = 0.09$ n = 2

Find F.

$F = P(1+j)^n = 300{,}000(1+0.09)^2 = 356{,}430$

The amount to be paid is P356,430.

2. If a car loan of P790,000 requires a 20% downpayment. How much is the mortgage?

Solution.

Down payment = down payment rate x cash price
= 0.20(790,000)
= 158,000
Amount of the Loan = cash price – down payment
= 790,000 – 158,000
= 632,000

The mortgage is P632,000.

3. A person borrowed P1,000,000 for the purchase of a car. If his monthly payment is P25,000 on an 8-year mortgage, find the total amount of interest.

Solution.

The total amount paid is given by
Total Amount = (25,000)(12 months)(8 years)
= 2,400,000

Thus, the total interest is the difference between the total amount paid and the amount of the mortgage;
Total Interest = 2,400,000 – 1,000,000
= 1,400,000

The total interest is P1,400,000.

4. A consumer loan worth P80,000 is to be repaid in 12 months at 8% convertible monthly. How much is the monthly payment?***Solution.***

Given: $A = 80{,}000$ $i^{(12)} = 0.08$ $j = \frac{i^{(12)}}{12} = \frac{0.08}{12}$ n=12

Find: the regular payment R

$$R = \frac{P}{\left[\frac{1-(1+j)^{-n}}{n}\right]} = \frac{80{,}000}{\left[\frac{1-(1+\frac{.08}{12})^{-12}}{\frac{.08}{12}}\right]} = 6{,}959.07$$

The monthly payment is P6,959.07.

5. A business loan worth P250,000 is to repay in quarterly installment for 1 year. How much is the quarterly payment if money is worth 8% converted quarterly?

Solution.

Given : $P = 250{,}000$ $i^{(4)} = 0.08$ $j = \frac{i^{(4)}}{4} = \frac{0.08}{4} = 0.02$ n=4

Find: the regular payment R

$$R = \frac{P}{\left[\frac{1-(1+j)^{-n}}{n}\right]} = \frac{250{,}000}{\left[\frac{1-(1+0.02)^{-4}}{0.02}\right]} = 65{,}655.94$$

The quarterly payment is P65,655.94.

6. Mr. Baldonado is considering to pay his outstanding balance after 6 years of payment. The original amount of the loan is P500,000 payable annually in 10 years. If the interest rate is 10% per annum and the regular payment is P81,372.70 annually, how much is the outstanding balance after the 6^{th}payment?

Solution.

Given: P =500,000 R = 81,372.70

$j = 0.10$ n=10 k=6

Find: outstanding balance after 6 payments (or present value of the remaining 4 payments)

$$B_6 = R\left[\frac{1-(1+j)^{-4}}{i}\right] = 81{,}372.70\left[\frac{1-(1+0.10)^{-4}}{0.10}\right] = 257{,}940.51$$

The outstanding balance after the 6^{th} payment is P257,940.51.

7. Mrs. Tan got a business loan worth P800,000. She promised to pay the loan semi-annually in 5 years. The semi-annual payment is P103,603.66 if money is worth 10% converted semi-annually. How much is the outstanding balance after the third payment?

Solution.

Given: A =800,000 R = 103,603.66 $i^{(2)} = 0.10$ $j = \frac{i^{(2)}}{2} = 0.05$

n= 10 k=3

Find: outstanding balance after 3 payments (or present value of the remaining 7 payments)

$$B_3 = R\left[\frac{1-(1+j)^{-7}}{j}\right] = 103603.66\left[\frac{1-(1.05)^{-7}}{.05}\right] = 599{,}489.46$$

The outstanding balance after the 3rd payment is P599,489.46.

8. Alana and her family acquired a loan amounting to P2,000,000. Her monthly amortization is P21,064.48 for 25 years. The interest rate is 12% convertible monthly. Find the amount of interest and the amount of principal paid on the 121st payment.

Solution.

Given: P =2,000,000 R = 21,064.48 $i^{(12)} = 0.12$ $j = \frac{i^{(12)}}{12} = \frac{0.12}{12} = 0.01$

n = 300 (monthly for 25 years) k=120 (120 payments have been made already)
Find: interest and principal components of the 121st payment

We first need to find the outstanding balance after 120 payments. After the 120th payment, there will be 180 remaining payments.

$$B_{120} = R\left[\frac{1-(1+i)^{-180}}{i}\right] = 21064.48\left[\frac{1-(1+0.01)^{-180}}{0.01}\right] = 1{,}755{,}127.53$$

$$I_{121} = i\ (P_{50}) = (0.01)(1{,}755{,}127.53) = 17{,}551.28$$
$$PR_{121} = R - I_{50} = 21{,}064.48 - 17{,}551.28 = 3{,}513.20$$

Thus, for the 121st payment, the part that goes to pay the interest is P17,551.28 and the part that goes to pay the principal is P3,513.20.

9. A loan amounting to P10,000 is to be paid annually for 4 years with an interest rate of 5% compounded annually. The annual amortization is P2,820.11. Complete the following table, and be guided by the questions below.

Period	**Regular Payment R**	**Interest Component of Payment**	**Principal Component of Payment**	**Outstanding Balance**
0				**A**
1	**B**	**500**	**2,320.11**	**7,679.56**
2	**2,820.11**	**F**	**G**	**5,243.54**
3	**2,820.11**	**262.18**	**2,557.93**	**H**
4	**2,820.11**	**134.29**	**2685.82**	**I**
TOTALS	**C**	**D**	**E**	

A. How much is the amount of the loan? (outstanding balance at period 0)

B. How much is the first annual payment?

C. How much is the total amount of payment?

D. How much is the total interest paid?

E. How much is the total payments for the principal?

F. For the second payment, how much goes to pay the interest?

G. For the second payment, how much goes to pay the principal?

H How much is the outstanding balance after the 3^{rd} payment?

I. How much should be the entry in the outstanding balance after the last payment

Solution.

A. The amount of the loan is P10,000, and this is the outstanding balance at time 0.

B. All payments are the same, so the answer isP2,820.11.

C. The total amount paid is (4)(P2820.11)= P11,280.44.

D. The total interest paid is = Total paid – Total payment

= 11,280.44 –10,000=P1,280.44

E. The total payments for the principal must be equal to the loan amount P10,000.

F. The outstanding balance after one payment is given in the table(P7,679.56). This amount will be charged 5% interest:

$$I_2 = i\ (P_1) = (0.05)(7{,}679.56) = P383.98$$

G. The amount of the second payment that goes to pay the principal is

$$PR_2 = R - I_2 = 2{,}820.11 - 383.98 = P2{,}436.13$$

H. Since, after the third payment, there is only 1 remaining payment. The outstanding balance after the third payment is given by

$$P_3 = R\left[\frac{1-(1+j)^{-(n-k)}}{j}\right] = 2820.11\left[\frac{1-(1+0.05)^{-1}}{0.05}\right] = 2685.82$$

I. Since all payments are made already, the outstanding balance should be 0.

Lesson 35 Supplementary Exercises

1. A business loan of P1,000,000 is to be repaid in full after 3 years. If the interest rate is 7% per annum. How much should be paid after 3 years?

2. Mr. Espiritu obtained a P470,000 car loan that will be paid in full after 2 1/2 years. If interest is at 12% compounded every 6 months, how much is to be paid after 2 ½ years?

3. Ms. Newman had a P55,000 business loan that is to be paid in 8 months. If interest rate is 12% compounded monthly, how much should be paid after 8 months?

4. A consumer loan of P80,000 is obtained that is due 3 months from now. If interest rate is 8% compounded quarterly, how much should be paid?

5. For the purchase of an SUV worth P1,200,000, the bank requires a minimum amount of 20% down payment, find the mortgaged amount.

6. For the purchase of a farm , the bank requires a 30% down payment . How much is the mortgaged amount if the cash value of the farm is P3,500,000?

7. Suppose that a condominium unit is purchased for P2,800,000 and the bank requires 30% down payment, how much is the mortgaged amount?

8. A house and lot has a cash value of P600,000. The bank offers a minimum amount of 25% down payment. How much is the loan or the mortgaged amount?

9. A family obtained a P4,500,000 mortgage. If the monthly payment is P50,000 for 12 years, how much is the total interest paid?

10. For a purchase of a P700,000 car, the monthly amortization is P20,500 for 4 years. How much is the total interest?

11. Ms. Santos obtained a condominium unit loan worth P2,700,000. If the monthly payment is P29,100 for 15 years, how much is the total interest?

12. Mr. Morales obtained a 5-year mortgage for P1,700,000. If his monthly payment is P47,500 , how much is the total interest?

13. A consumer loan worth P75,000 is to be repaid in 18 months at 12% convertible monthly. How much is the monthly payment?

14. Mr. Oclarit got a P90,000 loan to be repaid semi-annually in 3 ½ years. If interest rate is 14% compounded semi-annually, how much is the semi-annual payment?

15. A loan of P50,000 is to be amortized by paying quarterly in 1 year. If money is worth 10% compounded quarterly, how much is the monthly installment?

16. Mr and Mrs Avila decided to purchase a P5,000,000 house and lot. After deducting the downpayment, the mortgage amount is P4,000,000. If interest is at

10% compounded monthly, how much is the monthly installment if they plan to amortize the loan in 20 years?

17. Mr and Mrs Ramos had a housing loan payable monthly for 25 years. After paying for 12 years, how many payments are left?

18. Mr. Ramos got a business loan worth P100,000 payable quarterly for 3 years. In 1 ½ years, how many payments are left?

19. Ms. Castillo got a business loan worth P850,000. She promised to pay the loan quarterly in 4 years. The quarterly payment is P65,109.14 if money is worth 10% converted quarterly. How much is the outstanding balance after the second year?

20. A loan is being amortized by paying P12,000 monthly for 36 months. If money is worth 9% compounded monthly, how much is the outstanding balance after 12 payments?

21. Gabby is paying P1,000 monthly for the payment of his loan for 2 years now. At the moment, he still has 6 remaining payments. How much is the outstanding now if $i^{(12)} = 0.06$?

22. Ms. Lim pays P3,500 quarterly for a loan for 1 year now. She still needs to pay quarterly for another 3 years. How much is the outstanding balance now, if $i^{(4)} = 0.12$?

23. A consumer loan has a scheduled payment of P2,000 every quarter for 3 years. If money is worth 9% compounded quarterly, how much of the 5th payment goes to pay the interest?

24. A business loan is to be amortized monthly by paying P10,000 in 4 years. How much is the interest paid on the last payment if interest rate is 12% monthly?

25. Mr. Bainto has a loan that is to be amortized by paying monthly payments of P3,200 for 1 year. After paying for 6 months, he decided to pay off the loan. How much of the 6th payment goes to pay the principal if money is worth 12% compounded monthly?

26. Ms.Lachica got a car loan that requires a monthly payment of P13,000 for 5 years. She plans to pay off the loan after paying for 3 years. How much of the 13th payment goes to pay the principal if the interest rate is 10% compounded monthly?

27. Study the amortization schedule and fill in the blanks.

28. A loan amounting to P100,000 is to be paid annually for 4 years with an interest rate of 10% per annum. The annual amortization is P31,547.08.

Period	Regular Payment R	Interest Component of Payment	Principal Component of Payment	Outstanding Balance
0				**A**
1	**B**	10,000	21,547.08	78,452.92
2	31,547.08	**C**	**D**	54,751.13
3	31,547.08	5,475.11	26,071.97	28,679.16
4	31,547.08	2,867.92	28,679.16	**E**
TOTALS				

A. How much is the amount of the loan?
B. How much is the payment on the first period?
C. For the second payment, how much goes to pay the interest?
D. For the second payment, how much goes to pay the principal?
E. How much is the outstanding balance after the 4th payment?

Topic Test 1:

Answer the following problems completely. **[5 POINTS each]**

1. How much is the dividend for 1,000 shares of common stock at a par value of P20 if the dividend percentage is 2%?

2. If the declared dividend is P50,000,000 and if there are a total of 500,000 shares of common stock, how much is the dividend per share?

3. A financial institution declared a 2.5% dividend on a stock with a par value of P800. Mrs Lingan owns 1000 shares of stock with a par value of P800. How much is the dividend she received?

4. A bank showed that for a certain trading day, its opening price for its common stock is P 60 while its closing price is P 62.5. What is the percent change on this trading day?
5. If the annual earnings per share in stocks (dividend) of a certain company is P80 and yesterday's closing price is P130, what is the price-to-earnings ratio?
6. What is the current yield of a bond with a face value of P50,000 that pays coupons at 5% converted semi-annually and bought at face value?

7. What is the current yield of a bond with a face value of P200,000 that pays coupons at 4% converted semi-annually and bought at P 210,000?

8. What is the amount of the semi-annual coupon for a bond with a face value of P100,000 that pays 8% convertible semi-annually for its coupons?

For nos. 9-12, given the following listing on stocks, answer the questions that follow:

52 weeks								
HI	LO	STOCK	DIV	YLD%	PE	VOL(100s)	CLOSE	NETCHG
50	35.8	AAB	.40	1.2	10	2000	57.29	1.3
43.5	37	BBA	.35	1.9	5.7	1200	40.70	-0.5

9. What was the dividend per share last year for stock AAB?

10. What was the annual percentage yield last year for stock BBA?

11. What was the closing price in the last trading day for stock BBA?

12. For stock AAB, what was the closing price the day before the last trading day?

For nos. 13-16, refer to the following listing on bonds:

52 weeks								
				WEEKLY				
HIGH	LOW	NAME	CUR YLD	SALES (1000)	HIGH	LOW	LAST	NET CHG
101	88	XXY 5 21	3.1	20	100	98	100	-1
104	100	YYX 4 22	27	30	102	97	101	1

13. What is the current yield of the bond XXY?

14. What is the maturity year of the bond YYX?

15. What is closing price of the bond XXY in the last trading day?

16. For a P1,000 YYX bond ,what is the closing price last week?

17. If a house is sold for P5,000,000 and the bank requires 20% down payment, find the amount of the mortgage.

For nos 18-20, refer to the following problem: A loan of P1,000 is repaid by paying P263.80 annually for 5 years. If interest rate is 10% annually,

18. How much of the 1st payment goes to pay the interest?

19. How much of the 1^{st} payment goes to pay the principal?

20. After 4 years, how much is the pay off for the loan?

Topic Test 2

Answer the following problems completely.

1. What is the current yield of a bond with a face value of P90,000 that pays coupons at 6% converted semi-annually and bought at **[15]**

a) face value?

b) at a discount worth P 88,000?

c) at a premium worth P 95,000?

2. What is the amount of the semi-annual coupon for a bond with a face value of P30,000 that pays 6% convertible semi-annually for its coupons? **[5]**

3. How much is the dividend for 5,000 shares of common stock at a par value of P200 if the dividend percentage is 5%? **[5]**

4. If the declared dividend is P20,000,000 and if there are a total of 100,000 shares of common stock, how much is the dividend per share? **[5]**

5. A realty declared a 5% dividend on a stock with a par value of P3000. Mrs Abad owns 2000 shares of stock with a par value of P3000. How much is the dividend she received? **[5]**

6. A financial institution showed that for a certain trading day, its opening price for its common stock is P 105 while its closing price is P 98. What is the percent change on this trading day? **[5]**

7. If the annual earnings per share in stocks (dividend) of a certain company is P78 and yesterday's closing price is P95, what is the price-to-earnings ratio? **[5]**

8. A loan is to be amortized by paying 5000 annually for 7 years. If interest rate is 6% annually, **[15]**

a) how much is the outstanding balance after 5 payments?

b) how much of the 6^{th} payment goes to pay the interest?

c) how much of the 6^{th} payment goes to pay the principal?

Lesson 36: Propositions

Learning Outcome(s): At the end of the lesson, the learner is able to illustrate a proposition, symbolize propositions, and distinguish between simple and compound propositions

Lesson Outline:

1. Define proposition. Give examples and non-examples
2. Define simple and compound propositions. Distinguish simple and compound propositions.
3. Group work.

Definition: A **proposition** is a declarative sentence that is either true or false, but not both. If a proposition is true, then its truth value is true, which is denoted by T; otherwise, its truth value is false, which is denoted by F.

Propositions are usually denoted by small letters. For example, the proposition

p: Everyone should study logic

may be read as

p is the proposition "Everyone should study logic."

If a sequence of propositions is considered, we denote the propositions by $p_1, p_2, \ldots$

Example 1. Determine whether each of the following statements is a proposition or not. If it is a proposition, give its truth value.

p: Mindanao is an island in the Philippines.

q: Find a number which divides your age.

r: My seatmate will get a perfect score in the logic exam.

s: Welcome to the Philippines!

t: $3 + 2 = 5$

u: $f(x) = \frac{\sqrt{x}}{(x+1)}$ is a rational function.

v: What is the domain of the function?

w: I am lying.

p_1: It is not the case that $\sqrt{2}$ is a rational number.

p_2: Either logic is fun and interesting, or it is boring.

p_3: If you are a Grade 11 student, then you are a Filipino.

p_4: If you are more than 60 years old, then you are entitled to a Senior Citizen's card, and if you are entitled to a Senior Citizen's card, then you are more than 60 years old.

Solution. Recall that for a statement to be a proposition, it must be a declarative sentence, and it should have a truth value of either true or false, but not both true and false at the same time.

p. This is a declarative sentence, and Mindanao is an island in the Philippines. Hence p is a proposition.

q. This is an imperative sentence, and so it is not a proposition.

r. The statement is a declarative sentence, but its truth value will only be known after the logic exam. Nonetheless, it can either be true or false, but not both. Hence r is a proposition.

s. This statement is exclamatory, and hence it is not a proposition.

t. The given equation is a mathematical sentence. Translated into English, the equation reads "the sum of three and two is five", which is a declarative sentence. It is also a true statement. Hence t is a true proposition.

u. This is a declarative statement. Since the numerator of the function is not a polynomial function, the function f is not rational, so the statement is false. It is therefore a false proposition.

v. This is an interrogative sentence. Hence it is not a proposition.

w. Although w is a declarative sentence, it is not a proposition because it can neither be true nor false. Suppose w is true—that is, it is the case that I am lying. Since I am lying, my statement is not true, and so w must be false. The same conclusion can be drawn if w is assumed to be false.

p_1. While this is a declarative sentence, it can be shown that $\sqrt{2}$ cannot be expressed as a quotient of two integers. Thus, p_1 is a false proposition.

p_2. This is a declarative sentence, which is true, since students may find logic either fun or boring. Hence p_1 is a true proposition.

p_3. This is a declarative sentence, but it is not true. There are also Grade 11 students of other nationalities.

p_4. We know that it is a true proposition. This is also known as a biconditional statement, since we can rewrite it as "You are more than 60 years old if and only if you are entitled to a Senior Citizen's card."

Definition. A **compound proposition** is a proposition formed from simpler proposition using **logical connectors** or some combination of logical connectors. Some logical connectors involving propositions p and/or q may be expressed as follows:

not p

p and q

p or q

If p, then q

where <.> stands for some proposition. A proposition is **simple** if it cannot be broken down any further into other component propositions.

Example 2. For each of the propositions in Example 1, determine whether it is a simple or a compound proposition. If it is a compound proposition, identify the simple components.

Solution. The propositions p, r, t, and u are all simple propositions. On the other hand, the following are compound propositions:

p_1. **It is not the case that** $\sqrt{2}$ is a rational number.

p_2. Either logic is fun and interesting, **or** it is boring.

p_3. **If** you study hard, **then** you will get good grades.

p_4. **If** you are more than 60 years old, **then** you are entitled to a Senior Citizen's card, **and if** you are entitled to a Senior Citizen's card, **then** you are more than 60 years old.

Furthermore, we can determine the simple propositions that make up the propositions p_1, p_2, p_3, and p_4. We do so in the following table

Proposition	**Simple Component/s**
p_1	r: $\sqrt{2}$ is a rational number
p_2	f: Logic is fun i: Logic is interesting b: Logic is boring
p_3	h: You study hard g: You get good grades
p_4	a: You are more than 60 years old s: You are entitled to a Senior Citizen's card

The compound propositions can thus be expressed as follows:

p_1; **not** r

p_2: f **and** i **or** b

p_3: **if** h, **then** g

p_4: (**if** a **then** s) **and** (**if** s **then** a)

Solved Examples

1. Determine whether the following statements are propositions. If it is a proposition, determine its truth value if possible.

a: Is the traffic heavy along Katipunan Avenue today?

b: Please close the door.

c: $x + 2 = 11$.

d: 144 is a perfect square.

e: Keep right while passing through the corridor.

f: $2 \leq -1$.

g: The year 2016 is a leap year and the equation $x^2 + 1 = 0$ has no real solutions.

h: If a triangle has a right angle, then the triangle is called a right triangle.

i: Either a student takes a mathematics elective next semester, or he takes a business elective next year.

j: What is 5/10 in simplest form?

k: This is a false statement.

Solution.

a: This is not a proposition since it is a question.

b: This is not a proposition since it is an imperative statement.

c: While this is a proposition (in English, it can be read as "x plus 2 is equal to 11", which is declarative), its truth value cannot be ascertained since the value of x is unknown.

d: This is a true proposition, since it is declarative and we know that $12^2 = 144$.

e: This is an imperative statement, and hence it is not a proposition.

f: This is a proposition, since it can be translated as "2 is less than or equal to negative 1", but it is false.

g: This is a compound proposition, whose simple components are g_1: "The year 2016 is a leap year" and g_2: "The equation $x^2 + 1 = 0$ has no real solutions". Since both g_1 and g_1 are both true, then the compound proposition is true. The reason why the compound proposition is true will be explained in the next lesson.

h: This is the definition of a right triangle, and is hence a true proposition. Observe also that it is a compound proposition: **If** a triangle has a right angle, **then** the triangle is a called right triangle.

i: This is a compound proposition using the logical connector **or**. Its truth value cannot be determined since we do not know information about which elective the student took.

j: This is not a proposition, because it is a question.

k: While this is a declarative statement, it can neither be true nor false. If we suppose that the statement is false, then it consequently justifies that the statement is true. Hence, it is not a proposition.

2. For each of the following compound propositions, identify the simple components and the logical connectors used.

a: You went to the rock concert and your ears hurt.

b: It is not the case that it is Monday today.

c: Either today is a rainy day or today is a sunny day.

d: If fewer than 10 persons are in attendance, then the meeting will be cancelled.

e: Carlos will not fail the course if his final exam score exceeds 50%.

Solution.

a: For this compound proposition, we have the following simple components: a_1: "You went to the rock concert" and a_2: "Your ears hurt." The logical connector used with a_1 and a_2 is **and**.

b: The simple component here is b_1: "It is Monday today." The compound proposition is formed as b: **not**b_1.

c: Using the simple componentsc_1: "Today is a rainy day." and c_2: "Today is a sunny day.", the compound proposition c is formed with the logical connector **or**.

d: This compound proposition is of the if-then form; in particular, we have "**If** d_1, **then** d_2," where d_1:"Fewer than 10 persons are in attendance" and d_2: "The meeting will be cancelled."

e: This is also another if-then statement, but it is written in reverse order. The given statement may also be expressed as "If his final exam score exceeds 50%, then Carlos will not fail the course." Symbolically, we have "**If** e_1, **then not** e_2", where e_1: "His final exam score exceeds 50%" and e_2: "Carlos will fail the course."

Lesson 36 Supplementary Exercises

Determine whether the following statements are propositions. If the proposition is a compound proposition, identify the simple components and the logical connectors used.

1. Define a polynomial function.

2. Justin Bieber has over one million followers on Twitter and Instagram.

3. If Ted's score is less than 50, then Ted will fail the course.

4. What time is it?

5. Either it is sunny in Metro Manila or its streets are flooded.

6. Dinner is served with coffee or tea.

7. If a, b, and c denote the lengths of the legs and the hypotenuse of a right triangle, then $a^2 + b^2 = c^2$.

8. Timothy's average is at least 92 and he is getting an A for the course.

9. -5 is not a negative number.

10. A password must be at least 6 characters long or it must be at least 8 characters long.

11. If Jerry receives a scholarship, then he will go to college.

12. If you fix my computer, then I will pay you Php 2,000 and if I pay you Php 2,000, then you will fix my computer.

13. If you do not run 1 kilometer a day or do not eat properly, then you will not be healthy.

Lesson 37: Logical Operators

Learning Outcome(s): At the end of the lesson, the learner is able to perform different types of operations on propositions.

Lesson Outline:

1. Introduce how to construct a truth table.
2. Define the logical operators: negation, conjunction, disjunction, conditional and biconditional.
3. Class activity.

Definition: Given a proposition, its **truth table** show all its possible truth values.

Example 1. Since a proposition has two possible truth values, a proposition p would have the following truth table.

p
T
F

Truth tables can also be used to display various combinations of the truth values of two propositions p and q. The rows of the table will correspond to the each truth value combination of p and q, so there will be $2^2 = 4$ rows. The truth table for propositions p and q are as follows.

p	q
T	T
T	F
F	T
F	F

Similarly, suppose p, q, and r are propositions. Then the truth table involving the given propositions has $2^3 = 8$ rows, as shown below.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

In general, a truth table involving n propositions has 2^n rows.

Definition. The **negation** of a proposition p is denoted by

$$\sim p \text{ ("not } p\text{")}$$

and is defined through its truth table

p	$\sim p$
T	F
F	T

Example 2. State the negation of the following propositions.

n_1: $p(x) = (x-1)/(x+2)$ is a polynomial function.

n_2: 2 is an odd number.

n_3: The *tinikling* is the most difficult dance.

n_4: Everyone in Visayas speaks Cebuano.

Solution.

$\sim n_1$: "It is not true that $p(x) = (x-1)/(x+2)$ is a polynomial function" or simply "$p(x) = (x-1)/(x+2)$ is not a polynomial function".

$\sim n_2$: "It is not true that 2 is an odd number", or "2 is an even number."

$\sim n_3$: "The *tinikling* is not the most difficult dance."

$\sim n_4$: "Not everyone in Visayas speaks Cebuano."

Definition. The **conjunction** of the propositions p and q is denoted by

$$p \wedge q\text{: (}p\text{ and }q\text{)}$$

and is defined through its truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The propositions p and q are called **conjuncts**.

The conjunction $p \wedge q$ is true only when both conjuncts p and q are true, as shown in its truth table.

Example 3. Let p and q be the propositions

$$p\text{: Angels exist.}$$

$$q: \pi > 3.$$

Express the following conjunctions in English sentences or in symbols, as the case may be.

1. $p \wedge q$

2. $p \wedge (\sim q)$

3. "Angels do not exist and $\pi \leq 3$."

4. "While angels do not exist, $\pi > 3$."

***Solution*.**

1. "Angels exist and $\pi > 3$."

2. "Angels exist and $\pi \leq 3$" or "Angels exist, yet $\pi \leq 3$."

3. $(\sim p) \wedge (\sim q)$

4. $(\sim p) \wedge q$

Definition. The **disjunction** of two propositions p and q is denoted by

$$p \vee q: (p \text{ or } q)$$

and is defined through its truth table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The propositions p and q are called **disjuncts**.

The above truth table shows us that the disjunction $p \vee q$ is false only when both disjuncts p and q are false.

Example 4. Let p, q, and r be the following propositions:

p: Victor has a date with Liza.

q: Janree is sleeping.

r: Eumir is eating.

Express the following propositions in English sentences or in symbols, as the case may be.

1. $p \vee q$

2. $q \vee (\sim r)$

3. $p \vee (q \vee r)$

4. “Either Victor has a date with Liza or Janree is sleeping, or Eumir is eating.”

5. “Either Victor has a date with Liza and Janree is sleeping, or Eumir is eating.”

6. “Either Victor has a date with Liza, or Janree is sleeping, and Eumir is eating.”

7. “Either Victor has a date with Liza and Janree is sleeping, or Victor has a date with Liza and Eumir is eating.”

Solution. The corresponding English sentences or symbols are given below.

1. “Victor has a date with Liza or Janree is sleeping.”

2. “Either Janree is sleeping or Eumir is not eating.”

3. “Either Victor has a date with Liza, or Janree is sleeping, or Eumir is eating.”

4. $(p \vee q) \vee r$

5. $(p \wedge q) \vee r$

6. $p \vee (q \wedge r)$

7. $(p \wedge q) \vee (p \wedge r)$

It will be shown later that $p \vee (q \vee r)$ and $(p \vee q) \vee r$ are logically equivalent statements, so we can write $p \vee q \vee r$. Likewise, it will also be shown that $p \wedge (q \wedge r)$ and $(p \wedge q) \wedge r$ are logically equivalent, so we can write $p \wedge q \wedge r$.

Example 5. Suppose p, q, and r are the propositions defined above. Consider the scenario that one Friday night, Victor and Janree are studying for their Logic exam. Meanwhile, Eumir just tweeted a picture of himself eating crispy *pata* and *sisig*. What is the truth value of the proposition $(\sim p) \vee (q \wedge r)$?

Solution. From the given scenario, it follows that p and q are false proposition, and r is a true proposition. Therefore, the conjunction $q \wedge r$ is false. Also, $\sim p$ is true since p is false. Hence, the disjunction $(\sim p) \vee (q \wedge r)$ is true since at least one of its disjuncts is true.

The above discussion may also be summarized in the following table:

p	q	r	$\sim p$	$q \wedge r$	$(\sim p) \vee (q \wedge r)$
F	F	T	T	F	T

Definition. The **conditional** of the propositions p and q is denoted by

$$p \to q \text{: (If } p\text{, then } q\text{)}$$

and is defined through its truth table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The conditional $p \to q$ may also be read as "p implies q". The proposition p is called the **hypothesis,** while the proposition q is called the **conclusion**.

Example 6. Suppose that Geebee is a Grade 11 student. Consider the following conditionals.

p_1: If Geebee is in Grade 11, then she is a senior high school student.

p_2: If Geebee is in Grade 11, then she is working as a lawyer.

p_3: If Geebee has a degree in computer science, then she believes in true love.

Analyze the truth value of these conditionals.

Solution.

p_1: The hypothesis and the conclusion are true. Thus, p_1 is true (from the first row of the truth table for the conditional statement).

p_2: While the hypothesis is true, the conclusion is not (a Grade 11 student is not qualified to be a lawyer). From the definition of the conditional (second row of its truth table), the conditional statement p_2 is not true.

p_3: The hypothesis is not true since Geebee is still in Grade 11. On the other hand, we cannot determine the truth value of the conclusion "she believes in true love." From the last two rows of the truth table, regardless of the truth value of the conclusion, the conditional statement is true.

Example 7. One day, Richard tweeted: "If I get promoted, then I will stop posting selfies on Facebook." Let p be the statement "Richard gets promoted," and let q be the statement, "Richard stops posting selfies on Facebook." Determine whether the conditional $p \to q$ is true given the following scenarios.

Scenario A: Richard got the promotion and promptly stopped posting selfies on Facebook.

Scenario B: Richard got promoted, but then he realized that posting selfies has become a habit he cannot break easily.

Scenario C: Richard stopped posting selfies, but he did not get the promotion.

Solution.

Scenario A: Both the hypothesis and conclusion are true, so the conditional $p \rightarrow q$ is true.

Scenario B: In this scenario, the hypothesis p is true and conclusion q is false. By definition, the conditional $p \rightarrow q$ is false.

Scenario C: The hypothesis is false, hence regardless of the truth value of the conclusion, the conditional $p \rightarrow q$ is false. In this case, he did not break his promise since he was not promoted in the first place.

Example 8. Determine the truth values of the following propositions.

(a) If $2 > 0$, then there are 100 million Filipinos.

(b) If $2 > 0$, then there are only 5 languages in the Philippines.

(c) If $2 > 0$, then it is more fun in the Philippines.

Solution.

(a) The hypothesis and the conclusion are both true. Hence the conditional is true.

(b) The hypothesis is true, but the conclusion is wrong because there are more than 5 languages in the Philippines! In fact there are more than 100 languages in the country. Thus, the conditional is false.

(c) Because the hypothesis is false, the conditional is true whether it is indeed more fun in the Philippines or not.

Definition. The biconditional of propositions p and q is denoted by

$$p \leftrightarrow q: (p \text{ if and only if } q)$$

and is defined through its truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

The proposition may also be written as "p iff q". The propositions p and q are the **components** of the biconditional.

Example 9. Suppose that Geebee is a Grade 11 student. Let us now consider the following biconditionals.

p_1: Geebee is in Grade 11 if and only if she is a senior high school student.

p_2: Geebee is in Grade 11 if and only if she is working as a lawyer.

p_3: Geebee has a degree in Computer Science if and only if she believes in true love.

Solution.

p_1: Again, both simple components of p_1 are true. Therefore, the biconditional statement is true.

p_2: It is true that Geebee is in Grade 11, but it is not true that Geebee is working as a lawyer. Therefore, the biconditional is not true, referring to the second row of its truth table.

p_3: The first simple proposition, that Geebee has a degree in Computer Science, is false. The truth value of the entire biconditional depends on the truth value of the second simple component,that she believes in true love. If this is true, then the biconditional is false (referring to the third row of the truth table); otherwise, the biconditional is true.

Solved Examples

1. State the negation of the following propositions.

a: Mary received a text message from her friend.

b: $x^2 + y^2 = z^2$.

c: Eden teaches high-level mathematics to college students.

d: James was not able to fetch his sister from school today.

e: The student brings at most two bags with him every school day.

***Solution*.**

$\sim a$: Mary did not receive a text message from her friend.

$\sim b$: It is not the case that $x^2 + y^2 = z^2$ (or in symbols: $x^2 + y^2 \neq z^2$).

$\sim c$: Eden does not teach high-level mathematics to college students.

$\sim d$: James was able to fetch his sister from school today.

$\sim e$: The student brings more than two bags with him every school day.

2. Let p, q, and r be the propositions p: “Annie has a stomach ache.”, q: “Annie misses the exam.” and r: ”Annie receives a passing grade for the subject.” Express the following in English sentences or in symbols, as the case may be.

a. “Annie does not have a stomach ache, yet she misses the exam.”

b. “If Annie has a stomach ache, then she misses the exam and does not receive a passing grade for the subject.”

c. Either Annie has a stomach ache and misses the exam, or she does not miss the exam and she receives a passing grade for the subject.

d. $q \to (\sim r)$

e. $\big(p \to (\sim r)\big) \vee \big(q \to (\sim r)\big)$

f. $(\sim q) \leftrightarrow r$

Solution.

a. $(\sim p) \wedge q$

b. $p \to \big(q \wedge (\sim r)\big)$

c. $(p \wedge q) \vee \big((\sim q) \wedge r\big)$

d. "If Annie misses the exam, then she does not receive a passing grade for the subject."

e. "If Annie has a stomach ache, then she does not receive a passing grade for the subject, or if she misses the exam, then she does not receive a passing grade for the subject".

f. "Annie does not miss the exam if and only if she receives a passing grade for the subject."

3. Let u, v, and w be the propositions u: "Bea drives over the speed limit along the highway.", v: "Bea is pulled over by the traffic enforcer." and w: "Bea receives a speeding ticket." Express the following propositions in English sentences or in symbols, as the case may be.

a. "Bea does not drive over the speed limit and does not receive a speeding ticket."

b. "Whenever Bea drives over the speed limit, she is pulled over by the traffic enforcer."

c. "Bea receives a speeding ticket only if she drives over the speed limit along the highway or if she is pulled over by the traffic enforcer."

Note. Propositions of the form "p only if q" can be represented symbolically as $p \to q$. This follows from the definition of a biconditional statement.

d. $v \to \big((u \wedge w) \vee (\sim w)\big)$

e. $w \leftrightarrow (u \wedge v)$

f. $\big((\sim u) \wedge v\big) \to (\sim w)$

Solution.

a. $(\sim u) \wedge (\sim w)$

b. $u \rightarrow v$

c. $w \rightarrow (u \vee v)$

d. "If Bea is pulled over by the MMDA traffic enforcer, then either she drives over the speed limit and she receives a speeding ticket, or she does not receive a speeding ticket."

e. "Bea receives a speeding ticket if and only if she drives over the speed limit and she is pulled over by the MMDA traffic enforcer."

f. "If Bea does not drive over the speed limit and she is pulled over by the MMDA traffic enforcer, then she does not receive a speeding ticket."

4. Suppose p is a true proposition, q is a false proposition, and r is a true proposition. Determine the truth value of the following propositions.

a. $(p \wedge q) \wedge r$

b. $p \vee (q \vee r)$

c. $p \rightarrow \big((\sim q) \vee r\big)$

d. $(p \vee q) \wedge (\sim r)$

e. $(\sim p) \rightarrow (q \rightarrow r)$

f. $(p \leftrightarrow q) \vee ((\sim p) \leftrightarrow q)$

Solution.

a. Since q is false, then $p \wedge q$ is false. Thus, $(p \wedge q) \wedge r$ is false.

b. At least one of the disjuncts of $q \vee r$ is true, so the disjunction is true $q \vee r$. Therefore, $p \vee (q \vee r)$ is true.

c. Note that $\sim q$ is true since q is false. Therefore, $(\sim q) \vee r$ is true. Since both the hypothesis and the conclusion in the conditional is true, then the statement $p \sim \big((\sim q) \vee r\big)$ is true.

d. Since p is true, then $p \vee q$ is true. Since r is true, $\sim r$ is false. Therefore, $(p \vee q) \wedge (\sim r)$ is false.

e. Given that p is true, $\sim p$, which is the hypothesis, is false. Therefore, regardless of the truth value of $q \rightarrow r$, the conditional $(\sim p) \rightarrow (q \rightarrow r)$ is true.

f. Since p and q do not have the same truth value, $p \leftrightarrow q$ is false. Likewise, $(\sim p) \leftrightarrow q$ is true since $\sim p$ and q are both false. Therefore, the disjunction $(p \leftrightarrow q) \vee ((\sim p) \leftrightarrow q)$ is true.

5. Determine the truth values of the propositions p and q that will make the following statements false.

a. $\big(p \vee (\sim q)\big) \rightarrow q$

b. $(p \vee (\sim q)) \rightarrow (p \wedge q)$

Solution.

a. For this conditional to be false, $p \vee (\sim q)$ must be true and q must be false. Since q must be false, then $\sim q$ is true, and so the disjunction $p \vee (\sim q)$ is automatically true. Therefore, the conditional $\big(p \vee (\sim q)\big) \rightarrow q$ is false if and only if q is false and p has any truth value.

b. We require that $p \wedge q$ is false while $p \vee (\sim q)$ is true. For $p \wedge q$ to be false, at least one of p and q must be false. We take cases.

Suppose p is false. Then for $p \vee (\sim q)$ to be true, $\sim q$ must be true, or q must be false. Therefore, if p and q are both false, then $(p \vee (\sim q)) \rightarrow (p \wedge q)$ is false.

Suppose p is true. Then q must be false (for $p \wedge q$ to be false). If q is false, then $\sim q$ is true, and so $p \vee (\sim q)$ is true. Thus, if p is true and q is false, then $(p \vee (\sim q)) \rightarrow (p \wedge q)$ is false.

Therefore, we have two sets of truth values that will make the proposition true:

p	q
F	F
T	F

Lesson 37 Supplementary Exercises

1. State the negation of the following propositions.

m: Mathematics is easy to study.

n: Nellie's favorite song is "Hero" by Mariah Carey.

o: Logic is not taken up in junior high school.

$$p: x^2 - 4 \leq 3 + x$$

q: There are at least three people in the meeting room at the moment.

2. Let p, q, and r be the propositions p: "Rena eats at the Spanish restaurant.", q: "Rena orders the restaurant's special paella.", r: "Rena has dessert." Express the following statements in English sentences or in symbols, as the case may be.

a. "Rena orders the restaurant's special paella, but she does not have dessert."

b. "Whenever Rena eats at the Spanish restaurant, either she orders the restaurant's special paella or she has dessert."

c. "If Rena eats at the Spanish restaurant, she orders the restaurant's special paella if and only if she does not order dessert."

d. $p \to \big((q \wedge (\sim r)) \vee r\big)$

e. $p \to \big((\sim q) \to r\big)$

f. $\big((\sim q) \wedge (\sim r)\big) \to (\sim p)$

3. Consider the propositions u: "Pam works as a sales associate." v: "Bryan has a temporary position in the office." and w: "Pam and Bryan are co-workers." Express the following propositions in English sentences or in symbols as the case may be.

a. "Pam and Bryan are co-workers, but Pam works as a sales associate and Bryan does not have a temporary position in the office."

b. "Pam and Bryan are co-workers, but if Bryan has a temporary position in the office, then Pam does not work as a sales associate."

c. "Pam and Bryan are co-workers if and only if either Pam works as a sales associate or Bryan has a temporary position in the office."

d. $w \to (u \leftrightarrow v)$

e. $\big(u \leftrightarrow (\sim w)\big) \wedge \big((\sim v) \to w\big)$

f. $\big((\sim u) \vee (\sim v)\big) \to (\sim w)$

4. Suppose p and q are true propositions and r is a false proposition. Determine the truth value of the following compound propositions.

a. $(\sim p) \wedge \big(p \vee (\sim q)\big)$

b. $(\sim q) \vee ((\sim p) \wedge q)$

c. $p \to ((\sim q) \wedge r)$

d. $(\sim q) \to \big((\sim p) \leftrightarrow r\big)$

e. $(p \leftrightarrow q) \wedge ((\sim r) \to q)$

f. $\big((\sim p) \leftrightarrow (\sim q)\big) \leftrightarrow (p \leftrightarrow r)$

5. Determine the truth values of p and q that will make the following propositions false.

a. $p \vee \big((\sim q) \to p\big)$

b. $\big((p \to q) \wedge q\big) \to p$

Lesson 38: Truth Tables

Learning Outcome(s): At the end of the lesson, the learner is able to determine the possible truth value of a given compound proposition using truth tables and identify tautologies and contradictions.

Lesson Outline:

1. Constructing truth tables.
2. Define a tautology and contradiction.
3. Class activity.

Example 1. Let p and q be propositions. Construct the truth table for the compound proposition $(p \to q) \wedge (q \to p)$.

***Solution*.** Note that there are two propositions, p and q, involved in the compound proposition. Thus, the truth table will contain 4 rows, the first two columns of which are

p	q
T	T
T	F
F	T
F	F

Using the truth table for the definition of the conditional statements $p \to q$ and $q \to p$, we add two more columns to indicate the truth values of $p \to q$ and $q \to p$:

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

In the final column, we add the truth value of $(p \to q) \wedge (q \to p)$, which is a conjunction involving $p \to q$ and $q \to p$ as disjuncts.

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Example 2. Consider the compound proposition $[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$. Construct its truth table.

Solution. There are three primitive propositions involved, and so the truth table for the compound proposition has 8 rows.

We first consider the truth table pertaining to $(p \to r) \wedge (q \to r)$, the hypothesis of the entire conditional.

p	q	r	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
T	T	T	T	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Next we consider the conclusion $(p \vee q) \to r$ of the conditional. For this, we require the truth value of $p \vee q$ and $(p \vee q) \to r$, which will be appended to the table above.

p	q	r	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$	$p \vee q$	$(p \vee q) \to r$
T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

Lastly, we consider the truth value of the proposition s which we define to be $[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$

p	q	r	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$	$p \vee q$	$(p \vee q) \to r$	s
T	T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F	T
T	F	T	T	T	T	T	T	T
T	F	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F	T
F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	F	T	T

Note that regardless of the truth values of p, q, and r, proposition s is always true. Such propositions are called **tautologies.**

Definition. A proposition that is always true is called a **tautology**, while a proposition that is always false is called a **contradiction**. We denote tautologies by τ and contradictions by ϕ.

Example. Let p and q be propositions. Using truth tables, show the following:

a. $p \vee \tau$ is a tautology.

b. $p \wedge \phi$ is a contradiction

c. $p \rightarrow (p \vee q)$ is a tautology.

d. $(p \wedge (\sim q)) \wedge (p \wedge q)$ is a contradiction.

Solution.

a. Note that τ is always true. Hence in the disjunction $p \vee \tau$, there is at least one true disjunct. Therefore, $p \vee \tau$ is a tautology since regardless of the truth value of p, $p \vee \tau$ is true.

p	τ	$p \vee \tau$
T	T	T
F	T	T

b. Since ϕ is always false, then the second column of the truth table we will be constructing will contain Fs. We have the following truth table

p	ϕ	$p \wedge \phi$
T	F	F
F	F	F

Regardless of the truth value of p, $p \wedge \phi$ is always false. Hence, it is a contradiction.

c. We have the following truth table

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Regardless of the truth values of p and q, $p \rightarrow (p \vee q)$ is always true, so it is a tautology.

d. We have the following truth table

p	q	$\sim q$	$p \wedge (\sim q)$	$p \wedge q$	$(p \wedge (\sim q)) \wedge (p \wedge q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	F	F	F
F	F	T	F	F	F

We note that $(p \wedge (\sim q)) \wedge (p \wedge q)$ is false for any combination of truth values of p and q. Therefore, $(p \wedge (\sim q)) \wedge (p \wedge q)$ is a contradiction.

Solved Examples

1. Construct the truth table for the following compound propositions. Assume all variables denote propositions.

a. $(\sim p) \wedge (q \wedge (\sim r))$

b. $p \wedge \left[\left(q \vee (\sim p)\right) \wedge (\sim q)\right]$

c. $(p \rightarrow q) \leftrightarrow \left((\sim q) \rightarrow (\sim p)\right)$

d. $\left[(p \vee q) \vee \left((\sim p) \wedge q\right)\right] \rightarrow q$

e. $(p \rightarrow q) \wedge ((\sim p) \rightarrow r)$

Solution.

a.

p	q	r	$\sim p$	$\sim r$	$q \wedge (\sim r)$	$(\sim p) \wedge (q \wedge (\sim r))$
T	T	T	F	F	F	F
T	T	F	F	T	T	F
T	F	T	F	F	F	F
T	F	F	F	T	F	F
F	T	T	T	F	F	F
F	T	F	T	T	T	T
F	F	T	T	F	F	F
F	F	F	T	T	F	F

b. Let r denote the proposition $p \wedge \left[\left(q \vee (\sim p)\right) \wedge (\sim q)\right]$.

p	q	$\sim p$	$\sim q$	$q \vee (\sim p)$	$\left(q \vee (\sim p)\right) \wedge (\sim q)$	r
T	T	F	F	T	F	F
T	F	F	T	F	F	F
F	T	T	F	T	F	F
F	F	T	T	T	T	F

We note that $p \wedge \left[\left(q \vee (\sim p)\right) \wedge (\sim q)\right]$ is a contradiction.

c. Let r denote the proposition $(p \rightarrow q) \leftrightarrow \left((\sim q) \rightarrow (\sim p)\right)$.

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$(\sim q) \rightarrow (\sim p)$	r
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Hence, $(p \rightarrow q) \leftrightarrow \left((\sim q) \rightarrow (\sim p)\right)$ is a tautology.

d. Let r denote the proposition $[(p \vee q) \vee ((\sim p) \wedge q)] \rightarrow p$.

p	q	$\sim p$	$p \vee q$	$(\sim p) \wedge q$	$(p \vee q) \vee ((\sim p) \wedge q)$	r
T	T	F	T	F	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	F
F	F	T	F	F	F	T

e. Let s denote the proposition $(p \rightarrow q) \wedge ((\sim p) \rightarrow r)$

p	q	r	$\sim p$	$p \rightarrow q$	$(\sim p) \rightarrow r$	s
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	F	T	F
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	F

2. Show that the following statements are tautologies by constructing the truth table for each.

a. $(p \rightarrow q) \rightarrow ((\sim p) \vee q)$

b. $p \rightarrow (q \leftrightarrow (p \wedge q))$

c. $p \vee [\sim (p \wedge q)]$

d. $(\sim (p \wedge q)) \rightarrow ((\sim p) \vee (\sim q))$

Solution.

a. Let r denote the proposition $(p \rightarrow q) \rightarrow ((\sim p) \vee q)$

p	q	$\sim p$	$p \rightarrow q$	$(\sim p) \vee q$	r
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Hence, $(p \rightarrow q) \rightarrow ((\sim p) \vee q)$ is a tautology.

b. Let r denote the proposition $p \rightarrow (q \leftrightarrow (p \wedge q))$.

p	q	$p \wedge q$	$q \leftrightarrow (p \wedge q)$	r
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	T	T

Hence, $p \to \big(q \leftrightarrow (p \wedge q)\big)$ is a tautology.

c.

p	q	$p \wedge q$	$\sim (p \wedge q)$	$p \vee [\sim (p \wedge q)]$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Hence, $p \vee [\sim (p \wedge q)]$ is a tautology.

d. Let r denote the proposition $\big(\sim (p \wedge q)\big) \to \big((\sim p) \vee (\sim q)\big)$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$(\sim p) \vee (\sim q)$	r
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

Therefore, $\big(\sim (p \wedge q)\big) \to \big((\sim p) \vee (\sim q)\big)$ is a tautology.

Lesson 38 Supplementary Exercises

1. Construct the truth table for the following compound propositions. Assume all variables denote propositions.

a. $(p \vee q) \wedge [\sim (p \wedge q)]$

b. $\sim \Big(p \to \big(q \to (p \wedge q)\big)\Big)$

c. $(p \leftrightarrow q) \wedge ((\sim p) \to q)$

d. $(p \to q) \vee \big((\sim p) \to (\sim r)\big)$

e. $(p \to (q \to r)) \to \big((p \wedge q) \to r\big)$

2. Show that the following statements are tautologies by constructing the truth table for each.

a. $\big((\sim p) \vee q\big) \to (p \to q)$

b. $\big((\sim q) \to (\sim p)\big) \to (p \to q)$

c. $[\sim (p \leftrightarrow q)] \leftrightarrow \big[(p \vee q) \wedge \big(\sim (p \wedge q)\big)\big]$

d. $[p \wedge (q \vee r)] \to [(p \wedge q) \vee (p \wedge r)]$

e. $[p \vee (q \wedge r)] \to [(p \vee q) \wedge (p \vee r)]$

Lesson 39: Logical Equivalence and Conditional Propositions

Learning Outcome(s): At the end of the lesson, the learner is able to identify logically equivalent propositions, and illustrate different forms of conditional propositions.

Lesson Outline:

1. Define a logical equivalence.
2. Define different forms of conditional proposition.
3. Seatwork.

Definition. Two propositions p and $q$are logically equivalent, denoted by $p \Leftrightarrow q$, if they have the same truth values for all possible truth values of their simple components.

Logical equivalence can also be expressed in terms of a biconditional statement. Two propositions p and q are logically equivalent if the proposition $p \leftrightarrow q$ is always true (or is a tautology).

Example 1. Show that $(p \to q) \Leftrightarrow [(\sim p) \vee q]$. We shall call this logical equivalence the *Switcheroo law*[19].

Solution. We need to show that $p \to q$ and $(\sim p) \vee q$ have the same truth values using a truth table.

p	q	$\boldsymbol{p \to q}$	$\sim p$	$(\sim \boldsymbol{p}) \vee \boldsymbol{q}$
T	T	**T**	F	**T**
T	F	**F**	F	**F**
F	T	**T**	T	**T**
F	F	**T**	T	**T**

Notice that the third and fifth columns of the truth table above contains the same truth values in the same sequence. Thus$(p \to q) \Leftrightarrow [(\sim p) \vee q]$

[19] Waner, C. & Costenoble, S.R. (2001). *Supplementary chapters to accompany Finite Mathematics*, 2nd ed. Brooks/Cole. (http://www.zweigmedia.com/RealWorld/logic/logicintro.html)

The table below shows the different logical equivalences that are used when manipulating compound propositions.

Theorem (Table of Logical Equivalences). Let p, q, and r be propositions. We have the following logical equivalences

Identity Laws	$(p \wedge \tau) \Leftrightarrow p$	$(p \vee \phi) \Leftrightarrow p$
Domination Laws	$(p \vee \tau) \Leftrightarrow \tau$	$(p \wedge \phi) \Leftrightarrow \phi$
Idempotent Laws	$(p \vee p) \Leftrightarrow p$	$(p \wedge p) \Leftrightarrow p$
Inverse Laws	$[p \vee (\sim p)] \Leftrightarrow \tau$	$[p \wedge (\sim p)] \Leftrightarrow \phi$
Double Negation	$\sim (\sim p) \Leftrightarrow p$	
Associative Laws	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
Commutative Laws	$p \vee q \Leftrightarrow q \vee p$	$p \wedge q \Leftrightarrow q \wedge p$
Distributive Laws	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
De Morgan's Laws	$\sim (p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$	$\sim (p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$
Absorption Laws	$p \vee (p \wedge q) \Leftrightarrow p$	$p \wedge (p \vee q) \Leftrightarrow p$

Note: In the previous lecture, it was shown that $p \leftrightarrow q$ has the same truth table as $(p \to q) \wedge (q \to p)$. Therefore, we can say that

$$p \leftrightarrow q \Leftrightarrow [(p \to q) \wedge (q \to p)].$$

Example 2. Show that $\sim (p \to q) \Leftrightarrow [p \wedge (\sim q)]$ using logical equivalences.

Solution. One way to do this is to construct a truth table for each logical expression then show that they have the same truth values. Another method is to use the logical equivalences to transform $\sim (p \to q)$ into $p \wedge (\sim q)$.

		Reason
	$\sim (p \to q)$	Given
$\Leftrightarrow$	$\sim ((\sim p) \vee q)$	Switcheroo
$\Leftrightarrow$	$\sim (\sim p) \wedge (\sim q)$	De Morgan's Law
$\Leftrightarrow$	$p \wedge (\sim q)$	Double Negation

Example 3. Let p and q be propositions. Construct the truth tables for each of the following conditionals: $p \to q$, $q \to p$, $(\sim p) \to (\sim q)$, $(\sim q) \to (\sim p)$.

Solution. We construct a single truth table containing each of the conditionals:

p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$(\sim p) \to (\sim q)$	$(\sim q) \to (\sim p)$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Note that the fifth and eighth columns are the same, so we have shown that $(p \to q) \Leftrightarrow [(\sim q) \to (\sim p)]$. Likewise, since the sixth and seventh columns are identical, it follows that $(q \to p) \Leftrightarrow [(\sim p) \to (\sim q)]$.

The conditionals that we considered in the previous example are the different forms of conditional propositions.

Definition. Suppose p and q are propositions. From the conditional proposition $p \to q$, we derive three other conditional statements, namely its

a. **Converse:** $q \to p$
b. **Contrapositive:** $(\sim q) \to (\sim p)$
c. **Inverse:** $(\sim p) \to (\sim q)$

Note that a conditional and the corresponding contrapositive are logically equivalent. Likewise for a given conditional statement, its converse and inverse are logically equivalent.

We can also show that $(p \to q) \Leftrightarrow [(\sim q) \to (\sim p)]$ using logical equivalences in the following manner.

		Reason
	$p \to q$	
$\Leftrightarrow$	$(\sim p) \vee q$	Switcheroo
$\Leftrightarrow$	$q \vee (\sim p)$	Commutative Law
$\Leftrightarrow$	$\sim(\sim q) \vee (\sim p)$	Double Negation
$\Leftrightarrow$	$(\sim q) \to (\sim p)$	Switcheroo

Likewise, we can also show that $(q \to p) \Leftrightarrow [(\sim p) \to (\sim q)]$ by means of logical equivalences:

		Reason
	$q \to p$	
$\Leftrightarrow$	$(\sim q) \vee p$	Switcheroo
$\Leftrightarrow$	$p \vee (\sim q)$	Commutative Law
$\Leftrightarrow$	$\sim(\sim p) \vee (\sim q)$	Double Negation
$\Leftrightarrow$	$(\sim p) \to (\sim q)$	Switcheroo

Example 4. Consider the following true conditional

$p \to q$: "If Geebee is in Grade 11, then she is a senior high school student."

State its (a) converse, (b) contrapositive, and (c) inverse and determine whether each statement is also true.

Solution.

a. Converse ($q \to p$): "If Geebee is a senior high school student, then she is in Grade 11." This is not necessarily true.

b. Contrapositive $((\sim q) \rightarrow (\sim p))$: "If Geebee is not a senior high school student, then she is not in Grade 11." This is true.

c. Inverse $((\sim p) \rightarrow (\sim q))$: "If Geebee is not in Grade 11, then she is not a senior high school student." This is not necessarily true.

Example 5. Let $p \rightarrow q$ be the political slogan: "If there are no corrupt people, then there are no poor people." State the converse, contrapositive, and the inverse of $p \rightarrow q$.

Solution.

Converse $(q \rightarrow p)$: "If there are no poor people, then there are no corrupt people."

Contrapositive $((\sim q) \rightarrow (\sim p))$: "If there are poor people, then there are corrupt people."

Inverse $((\sim p) \rightarrow (\sim q))$: "If there are corrupt people, then there are poor people."

Solved Examples

1. Determine the converse, contrapositive and the inverse of the following conditional propositions.

a. "If it is consumed in large volumes, then chocolate can be harmful to one's health."

b. "Whenever she will be given the chance to perform on stage, Whitney will sing all her classic songs."

c. "If Sheila will decide to enter through the front door, then she will be greeted by a large group of press people."

d. "Dana studied for the exam alone if her boyfriend decided to go home early to sleep."

e. "If at least one student slept in class, then Lyn gave a difficult quiz to wake the students."

Solution.

a. Note that the given proposition is of the form $p \rightarrow q$, where p: "It (chocolate) is consumed in large volumes" and q: "Chocolate can be harmful to one's health."

Converse $q \rightarrow p$	"If chocolate can be harmful to one's health, then it is consumed in large volumes."
Contrapositive $(\sim q) \rightarrow (\sim p)$	"If chocolate cannot be harmful to one's health, then it is not consumed in large quantities."
Inverse $(\sim p) \rightarrow (\sim q)$	"If it is not consumed in large volumes, then chocolate cannot be harmful to one's health."

b. We can write the given proposition as, "If she will be given the chance to perform on stage, then Whitney will sing her classic songs." It is now of the form $p \rightarrow q$, where p: "She will be given the chance to perform on stage." and q: "Whitney will sing her classic songs."

Converse $q \rightarrow p$	"If Whitney will sing her classic songs, then she will be given the chance to perform on stage."
Contrapositive $(\sim q) \rightarrow (\sim p)$	"If Whitney will not sing her classic songs, then she will not be given the chance to perform on stage."
Inverse $(\sim p) \rightarrow (\sim q)$	"If she will not be given the chance to perform on stage, then Whitney will not sing her classic songs."

c. The given conditional is of the form $p \rightarrow q$, where p: "Sheila will decide to enter through the front door" and q: "she will be greeted by a large group of press people".

Converse $q \rightarrow p$	"If she will be greeted by a large group of press people, then Sheila will decide to enter through the front door."
Contrapositive $(\sim q) \rightarrow (\sim p)$	"If she will not be greeted by a large group of press people, then Sheila will not decide to enter through the front door."
Inverse $(\sim p) \rightarrow (\sim q)$	"If Sheila will not decide to enter through the front door, then she will not be greeted by a large group of press people."

d. Note that we can write the given conditional as "If her boyfriend decided to go home early to sleep, then Dana studied for the exam alone," which is now of the form $p \rightarrow q$, where p: "Her boyfriend decided to go home early." and q: "Dana studied for the exam alone."

Converse $q \rightarrow p$	"If Dana studied for the exam alone, then her boyfriend decided to go home early to sleep."
Contrapositive $(\sim q) \rightarrow (\sim p)$	"If Dana did not study for the exam alone, then her boyfriend did not decide to go home early to sleep."
Inverse $(\sim p) \rightarrow (\sim q)$	"If her boyfriend did not decide to go home early to sleep, then Dana did not study for the exam alone."

e. The given proposition is of the form $p \rightarrow q$, where p: "At least one student slept in class" and q: "Lyn gave a difficult quiz to wake the students."

Converse $q \rightarrow p$	"If Lyn gave a difficult quiz to wake the students, then at least one student slept in the class."
Contrapositive $(\sim q) \rightarrow (\sim p)$	"If Lyn did not give a difficult quiz to wake the students, then no student slept in the class."
Inverse $(\sim p) \rightarrow (\sim q)$	"If no student slept in the class, then Lyn did not give a difficult quiz to wake the students."

2. Verify the following logical equivalences using (a) the known logical equivalences, and (b) truth tables.

a. $p \wedge q \Leftrightarrow \sim \big((\sim p) \vee (\sim q)\big)$

b. $(\sim p) \rightarrow (q \rightarrow r) \Leftrightarrow q \rightarrow (p \vee r)$

c. $p \leftrightarrow q \Leftrightarrow (\sim p) \leftrightarrow (\sim q)$

d. $\big(\sim (p \vee q)\big) \vee \big((\sim p) \wedge q\big) \Leftrightarrow \sim p$

Solution. We will verify these logical equivalences using the known logical equivalences. The construction of truth tables to prove the equivalence will be left to the student as an exercise.

a. We can start with the right-hand side of the equivalence.

		Reason
	$\sim [(\sim p) \vee (\sim q)]$	Given
$\Leftrightarrow$	$[\sim (\sim p)] \wedge [\sim (\sim q)]$	De Morgan's Law
$\Leftrightarrow$	$p \wedge q$	Double Negation

b.

		Reason
	$(\sim p) \rightarrow (q \rightarrow r)$	Given
$\Leftrightarrow$	$\big(\sim (\sim p)\big) \vee (q \rightarrow r)$	Switcheroo
$\Leftrightarrow$	$p \vee (q \rightarrow r)$	Double Negation
$\Leftrightarrow$	$p \vee [(\sim q) \vee r]$	Switcheroo
$\Leftrightarrow$	$(\sim q) \vee [p \vee r]$	Associative Law and Commutative Law
$\Leftrightarrow$	$q \rightarrow (p \vee r)$	Switcheroo

c. For this exercise, we will use the fact that conditionals are logically equivalent to their respective contrapositives.

		Reason
	$p \leftrightarrow q$	Given
$\Leftrightarrow$	$(p \rightarrow q) \wedge (q \rightarrow p)$	Logical equivalence for biconditionals
$\Leftrightarrow$	$[(\sim q) \rightarrow (\sim p)] \wedge [(\sim p) \rightarrow (\sim q)]$	Conditionals are logically equivalent to their contrapositives
$\Leftrightarrow$	$[(\sim p) \rightarrow (\sim q)] \wedge [(\sim q) \rightarrow (\sim p)]$	Commutative Law
$\Leftrightarrow$	$(\sim p) \leftrightarrow (\sim q)$	Logical equivalence for biconditionals

d.

		Reason
	$\left(\sim(p \vee q)\right) \vee \left((\sim p) \wedge q\right)$	Given
$\Leftrightarrow$	$[(\sim p) \wedge (\sim q)] \vee [(\sim p) \wedge q]$	De Morgan's Law
$\Leftrightarrow$	$\{[(\sim p) \wedge (\sim q)] \vee (\sim p)\} \wedge \{[(\sim p) \wedge (\sim q)] \vee q\}$	Distributive Law
$\Leftrightarrow$	$\{[(\sim p) \vee (\sim p)] \wedge [(\sim q) \vee (\sim p)]\} \wedge \{[(\sim p) \vee q] \wedge [(\sim q) \vee q]\}$	Distributive Law
$\Leftrightarrow$	$\{(\sim p) \wedge [(\sim q) \vee (\sim p)]\} \wedge \{[(\sim p) \vee q] \wedge \tau\}$	Idempotent Law and Inverse Law
$\Leftrightarrow$	$\{(\sim p) \wedge [(\sim q) \vee (\sim p)]\} \wedge [(\sim p) \vee q]$	Identity Law
$\Leftrightarrow$	$(\sim p) \wedge [(\sim p) \vee q]$	Absorption Law
$\Leftrightarrow$	$\sim p$	Absorption Law

Lesson 39 Supplementary Exercises

1. Determine the converse, contrapositive, and inverse of the following conditional propositions.

a. "If the clothes are neatly stacked and pressed, then the house help arrived today."

b. "If it did not flood yesterday, then the streets are dry today."

c. "Whenever Nico studied alone, he got the highest score in the class."

d. "Her parents gave her monetary allowance if Bianca accompanied her parents to the PTA meeting."

e. "If Mariah hit the high whistle note, then the audience gave her a standing ovation."

2. Verify the following logical equivalences using (a) the known logical equivalences and (b) truth tables.

a. $p \vee q \Leftrightarrow \sim((\sim p) \wedge (\sim q))$

b. $p \leftrightarrow q \Leftrightarrow \left[(p \wedge q) \vee \left((\sim p) \wedge (\sim q)\right)\right]$

c. $p \to q \Leftrightarrow \sim[p \wedge (\sim q)]$

d. $p \leftrightarrow (q \leftrightarrow q) \Leftrightarrow p$

Lesson 40: Valid Arguments and Fallacies

Learning Outcome(s): At the end of the lesson, the learner is able to illustrate different types of valid arguments and fallacies, and establish the validity of arguments.

Lesson Outline:

1. Introduction.
2. Define argument.
3. Define valid argument.
4. Define fallacy.
5. Define sound argument.

Definition. An **argument** is a compound proposition of the form

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q.$$

The propositions $p_1, p_2, \ldots, p_n$ are the **premises** of the argument, and q is the **conclusion**. Arguments can be written in **propositional form**, as in above, or in column or **standard form**:

$$\begin{array}{rl} & p_1 \\ & p_2 \\ & \vdots \\ & \underline{p_n} \\ \therefore & q \end{array}$$

Example 1. Explain why the following set of propositions is an argument.

If General Antonio Luna is a national hero, then he died at the hands of the Americans in 1899.

General Luna is a national hero.

Therefore, General Luna died at the hands of the Americans in 1899.

Solution. The set of propositions is an argument. The first two propositions are the premises of the argument, while the last proposition, marked by the word "therefore", is the conclusion of the argument.

Example 2. Write the following argument in propositional form and in standard form:

If there is a limited freshwater supply, then we should conserve water.

There is a limited freshwater supply.

Therefore, we should conserve water.

Solution. The premises of this argument are

p_1: If there is a limited freshwater supply, then we should conserve water.

p_2: There is a limited freshwater supply.

The conclusion is

q: We should conserve water.

In symbols, we can write the whole argument in propositional form $(p_1 \wedge p_2) \rightarrow q$ and in standard form

$$\begin{array}{rl} & p_1 \\ & \underline{p_2} \\ \therefore & q \end{array}$$

Example 3. Consider the arguments A and A' given below

A	$p \rightarrow q$	If my alarm sounds, then I will wake up.	A'	$p \rightarrow q$	If my alarm sounds, then I will wake up.
	p	My alarm sounded.		q	I woke up.
$\therefore$	q	Therefore, I woke up.	$\therefore$	p	Therefore, my alarm sounded.

We can analyze the arguments separately by looking at its *validity*.

Definition. An argument is **valid** if it satisfies the **validity condition:**

Is it logically impossible for the premises to be true and the conclusion false?

If the answer is affirmative, we say that the argument satisfies the validity condition, and is hence valid.

Solution. For argument A, we ask: can $p \rightarrow q$ and p be both true and q be false? To answer this, we look at the truth table for $p \rightarrow q$:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The highlighted row shows that both p and $p \rightarrow q$ are true if and only if q is true. Based on the truth table, it is not possible for p and $p \rightarrow q$ to be true and q to be false. Hence, argument A is valid.

For argument A', we ask: can $p \to q$ and q be both true and p be false? Looking at the same truth table,

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

the highlighted row shows that it is possible for $p \to q$ and q be both true and p be false. Hence, argument A' does not satisfy the validity condition and so it is not a valid argument.

Definition. A **valid argument** satisfies the validity condition; that is, the conclusion q is true whenever the premises $p_1, p_2, \ldots, p_n$ are all true. Alternatively, the argument is valid if the conditional

$$(p_1 \land p_2 \land \ldots \land p_n) \to q$$

is a tautology.

Example 4. Prove that the argument $[(p \to q) \land p] \to q$ is valid. This argument is known as **Modus Ponens** (or Rule of Detachment).

Solution. We only need to show that $[(p \to q) \land p] \to q$ is a tautology. We can do this by constructing a truth table.

p	q	$p \to q$	$(p \to q) \land p$	$[(p \to q) \land p] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	T	T

Since $[(p \to q) \land p] \to q$ is a tautology, then the argument is valid.

Example 5. Consider the arguments in the previous examples:

Argument A

If my alarm sounds, then I will wake up.

My alarm sounded.

Therefore, I woke up.

Argument B

If there is a limited freshwater supply, then we should conserve water.

There is a limited freshwater supply.

Therefore, we should conserve water.

Argument C

If General Antonio Luna is a national hero, then he died at the hands of the Americans in 1899.

General Luna is a national hero.

Therefore, General Luna died at the hands of the Americans in 1899.

Note that they are all of the form $[(p \to q) \wedge p] \to q$, or in standard form

$$\begin{array}{rc} & p \to q \\ & \underline{\quad p \quad} \\ \therefore & q \end{array}$$

Hence, by Modus Ponens, all three arguments are valid. **However, this does not mean that the conclusions are true**. Asserting that an argument is valid simply means that the conclusion **logically** follows from the premises.

Theorem (Rules of Inference). Let p, q, and r be propositions.

	Propositional Form	Standard Form
Rule of Simplification	$(p \wedge q) \to p$	$\begin{array}{rc} & \underline{p \wedge q} \\ \therefore & p \end{array}$
Rule of Addition	$p \to (p \vee q)$	$\begin{array}{rc} & \underline{p} \\ \therefore & p \vee q \end{array}$
Rule of Conjunction	$(p \wedge q) \to (p \wedge q)$	$\begin{array}{rc} & p \\ & \underline{q} \\ \therefore & p \wedge q \end{array}$
Modus Ponens	$[(p \to q) \wedge p] \to q$	$\begin{array}{rc} & p \to q \\ & \underline{p} \\ \therefore & q \end{array}$
Modus Tollens	$[(p \to q) \wedge (\sim q)] \to (\sim p)$	$\begin{array}{rc} & p \to q \\ & \underline{\sim q} \\ \therefore & \sim p \end{array}$
Law of Syllogism	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	$\begin{array}{rc} & p \to q \\ & \underline{q \to r} \\ \therefore & p \to r \end{array}$
Rule of Disjunctive Syllogism	$[(p \vee q) \wedge (\sim p)] \to q$	$\begin{array}{rc} & p \vee q \\ & \underline{\sim p} \\ \therefore & q \end{array}$
Rule of Contradiction	$[(\sim p) \to \phi] \to p$	$\begin{array}{rc} & \underline{(\sim p) \to \phi} \\ \therefore & p \end{array}$
Rule of Proof by Cases	$[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$	$\begin{array}{rc} & p \to r \\ & \underline{q \to r} \\ \therefore & (p \vee q) \to r \end{array}$

Example 6. Determine whether the following argument is valid:

If Antonio and Jose are friends, then they are Facebook friends.

Antonio and Jose are not Facebook friends.

Therefore, they are not friends.

Solution. Let p: "Antonio and Jose are friends." and q: "Antonio and Jose are Facebook friends." Then the given argument is of the form

$$\begin{array}{rl} & p \rightarrow q \\ & \underline{\sim q} \\ \therefore & \sim p \end{array}$$

This is valid by Modus Tollens.

Example 7. Determine which rule is the basis of each argument below.

a. Antonio Luna and Jose Rizal like Nelly Boustead.

Therefore, Antonio Luna likes Nelly Bousted.

b. Antonio Luna is a scientist.

Therefore, either Antonio Luna or Jose Rizal is a scientist.

c. If the Spaniards imprison Antonio Luna, then he will repent and not join the revolution.

If Antonio Luna regrets not joining the revolution, then he will go to Belgium to study the art of war.

Therefore, if the Spaniards imprison Antonio Luna, then he will go to Belgium to study the art of war.

Solution.

a. Let p: "Antonio Luna likes Nelly Boustead." and q: "Jose Rizal likes Nelly Boustead." The given argument is of the form

$$\begin{array}{rl} & \underline{p \wedge q} \\ \therefore & p \end{array}$$

By the Rule of Simplification, the argument is valid.

b. Let p: "Antonio Luna is a scientist." and q: "Jose Rizal is a scientist." The argument is of the form

$$\begin{array}{rl} & \underline{p} \\ \therefore & p \vee q \end{array}$$

By the Rule of Addition, the argument is valid.

c. Let p: "The Spaniards imprison Antonio Luna.", q: "Antonio Luna regrets not joining the revolution", and r: "Antonio Luna goes to Belgium to study the art of war."The argument is of the form

$$\begin{array}{rl} & p \to q \\ & \underline{q \to r} \\ \therefore & p \to r \end{array}$$

The argument is valid, by the Law of Syllogism.

Definition. An argument $(p_1 \wedge p_2 \wedge ... \wedge p_n) \to q$ which is not valid is called a **fallacy**. In a fallacy, it is possible for the premises $p_1, p_2, ..., p_n$ to be true, while the conclusion q is false. In this case, the conditional $(p_1 \wedge p_2 \wedge ... \wedge p_n) \to q$ is not a tautology.

Example 8. Prove that the argument $[(p \to q) \wedge q] \to p$ is a fallacy. This is known as the **Fallacy of the Converse**.

Solution. We show that $[(p \to q) \wedge q] \to p$ is a fallacy by means of a truth table.

p	q	$p \to q$	$(p \to q) \wedge q$	$[(p \to q) \wedge q] \to p$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	T	T

Since $[(p \to q) \wedge q] \to p$ is not a tautology, the argument is invalid and is hence a fallacy.

Note that it is sufficient to find truth values of p and q that will make the premises simultaneously true but the conclusion false. We call such set of values a **counterexample**. In this case, the counterexample is the case when p is false and q is true (see the third row of the truth table above).

Example 9. Show that the following arguments are fallacies.

A': If my alarm sounds, then I will wake up.

I woke up.

Therefore, my alarm sounded.

B': If there is a limited supply of freshwater, then I will conserve water.

I will conserve water.

Therefore, there is limited supply of freshwater.

Solution. Each of the arguments has the form

$$\begin{array}{rl} & p \to q \\ & \underline{\quad q \quad} \\ \therefore & p \end{array}$$

Similar to the previous example, each argument is a fallacy of the converse.

The following table lists some of the common fallacies in logic.

	Propositional Form	**Standard Form**
Fallacy of the Converse	$[(p \to q) \wedge q] \to p$	$p \to q$ q $\therefore p$
Fallacy of the Inverse	$[(p \to q) \wedge (\sim p)] \to (\sim q)$	$p \to q$ $\sim p$ $\therefore \sim q$
Affirming the Disjunct	$[(p \vee q) \wedge p] \to (\sim q)$	$p \vee q$ p $\therefore \sim q$
Fallacy of the Consequent	$(p \to q) \to (q \to p)$	$p \to q$ $\therefore q \to p$
Denying a Conjunct	$[\sim (p \wedge q) \wedge (\sim p)] \to q$	$\sim (p \wedge q)$ $\sim p$ $\therefore q$
Improper Transposition	$(p \to q) \to [(\sim p) \to (\sim q)]$	$p \to q$ $\therefore (\sim p) \to (\sim q)$

Example 10. Determine whether the given is a valid argument or a fallacy.

a. Either Alvin sings or dances with Nina.
   Alvin sang with Nina.
   Therefore, Alvin did not dance with Nina.

b. Either Alvin sings or dances with Nina.
   Alvin did not dance with Nina.
   Therefore, Alvin sang with Nina.

c. It is not true that Alvin sings and dances with Nina.
   Alvin did not sing with Nina.
   Therefore, Alvin danced with Nina.

Solution. Let p: "Alvin sings with Nina." and q: "Alvin dances with Nina."

a. The given argument is of the form

$$\begin{array}{rc} & p \vee q \\ & \underline{p} \\ \therefore & \sim q \end{array}$$

This is the fallacy of Affirming the Disjunct.

<u>Alternate Solution:</u> We can prove that the argument is not valid by finding a counterexample (i.e., truth values for p and q that make the propositional form of the tautology false). This happens when both p and q are false, as the following table shows.

p	q	$\sim q$	$p \vee q$	$(p \vee q) \wedge p$	$[(p \vee q) \wedge p] \rightarrow \sim q$
T	T	F	T	T	F

b. The given argument is of the form

$$\begin{array}{rc} & p \vee q \\ & \underline{\sim q} \\ \therefore & p \end{array}$$

The first premise can be written as $q \vee p$, by the Commutative Law, and so we can write

$$\begin{array}{rc} & q \vee p \\ & \underline{\sim q} \\ \therefore & p \end{array}$$

Which adheres to the Rule of Disjunctive Syllogism, the argument is valid.

c. In symbols, the argument is of the form

$$\begin{array}{rc} & \sim (p \vee q) \\ & \underline{\sim p} \\ \therefore & q \end{array}$$

This is the fallacy of Denying a Conjunct.

<u>Alternate Solution:</u> We can find a counterexample to show that the argument is not valid. If both p and q are false, then the propositional form of the argument is false.

p	$\sim p$	q	$(p \wedge q)$	$\sim (p \wedge q)$	$\sim (p \wedge q) \sim p$	$[\sim (p \wedge q) \sim p] \rightarrow q$
F	T	F	F	T	T	F

ENRICHMENT: Valid and Sound Arguments

Definition. An argument is said to satisfy the **truth condition** if its premises are generally true.[20]

Definition. A **sound argument** is a valid argument which also satisfies the truth condition. An argument which does not satisfy either the validity condition or the truth condition is called a **bad argument**.[21]

Example 11. The following arguments were already shown to be valid.

a. Antonio Luna and Jose Rizal like Nelly Boustead.

   Therefore, Antonio Luna likes Nelly Boustead.

b. Antonio Luna is a scientist.

   Therefore, either Antonio Luna or Jose Rizal is a scientist.

A simple history verification will show that the premises of both arguments are true. Nelly Boustead was the object of affection of Antonio Luna and Jose Rizal while they were in Spain.

Moreover, Luna is known as a brilliant general, but he is also a scientist. He studied Chemistry at the University of Santo Tomas, and went to Spain where he obtained his license and doctorate in pharmacy.

Hence, these arguments satisfy both truth and validity conditions, so they are sound arguments.

Example 12. Determine whether each of the following arguments is valid, and if each is sound.

a. If I was born poor, then I cannot serve my country.
   I was born poor.
   Therefore, I cannot serve my country.
b. If I study every day, then I will develop a good work ethic.
   I study every day
Therefore, I will develop a good work ethic.

Solution. By Modus Ponens, both arguments are valid. We then check for soundness through the truth condition.

a. Note that being poor does not prevent one from serving one's country (you can probably think of some examples). Hence, the given argument is a bad argument.

b. It is accepted as true that if one studies every day, then a good work ethic will be developed. However, it cannot be assumed true that "I study every day". If it is true, then the argument is sound. Otherwise, the argument is bad.

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[20] deLaplante, K. (2013). *What is a good argument? The truth condition* (https://www.youtube.com/watch?v=9mk8RWTsFFw).
[21] Ibid.

Solved Examples

1. Determine whether the following arguments are valid. If it is valid, then identify the rule of inference which justifies its validity. Otherwise, state a counterexample or identify the type of fallacy exhibited by the argument.

a. If it rains today, then $2 \times 2 = 4$. It rained today. Therefore, $2 \times 2 = 4$.

b. Either Lina or Lino will take the trash to the recycling center. Lina did not take the trash to the recycling center. Hence, Lino must have taken the trash to the recycling center.

c. If Joe makes a reviewer for his class and studies it well, then he will get a high grade in his exam. Joe did not get a high grade in his exam. Therefore, either Joe did not make a reviewer for his class or he did not study it well.

d. If f is a polynomial function, then it is also a rational function. Therefore, if f is a rational function, it is also a polynomial function.

e. If $x \geq 0$, then $x^2 \geq 0$. It holds that $x < 0$. Therefore, $x^2 < 0$.

f. It is not the case that x is an odd number and y is a prime number. Furthermore, x is not an odd number. Therefore, y is a prime number.

g. If Leona wins the singing competition, then she will land a recording contract with a famous company. She landed a recording contract with a famous company. It follows that Leona won the singing competition.

h. If Michael sleeps early tonight, then he will wake up early tomorrow. If he does not play with his pet dog, then he will wake up early tomorrow. Therefore if Michael sleeps early tonight or does not play with his pet dog, then he will wake up early tomorrow.

***Solution*.**

a. Let p: "It rains today" and q: $2 \times 2 = 4$. In proposition form, the argument assumes the form $[(p \to q) \wedge p] \to q$. Hence, the argument is valid by virtue of Modus Ponens.

b. Let p: "Lina will take the trash to the recycling center." and q: "Lino will take the trash to the recycling center." In propositional form, the argument is $[(p \vee q) \wedge (\sim p)] \to q$. Thus, the argument is valid by the Rule of Disjunctive Syllogism.

c. Let p: "Joe makes a reviewer for his class.", q: "Joe studies the reviewer well.", and r: "Joe will get a high grade in the exam."

In proposition form, the argument is $\{[(p \wedge q) \to r] \wedge (\sim r)\} \to [(\sim p) \vee (\sim q)]$. Note that the conclusion can also be written in the form $\sim (p \wedge q)$, and so the argument is

$$\{[(p \wedge q) \to r] \wedge (\sim r)\} \to \sim (p \wedge q)$$

This is valid by Modus Tollens.

d. Let p: "f is a polynomial function" and q: "f is a rational function." In standard form, the argument is

$$\begin{array}{rc} & p \rightarrow q \\ \hline \therefore & q \rightarrow p \end{array}$$

This is the Fallacy of the Consequent.

e. Let p: "$x \geq 0$" and q: "$x^2 \geq 0$". Then the argument is of the form

$$\begin{array}{rc} & p \rightarrow q \\ & \sim p \\ \hline \therefore & \sim q \end{array}$$

This is the Fallacy of the Inverse.

f. Let p: "x is an odd number," and q: "y is a prime number." The argument is of the form

$$\begin{array}{rc} & \sim (p \wedge q) \\ & \sim p \\ \hline \therefore & q \end{array}$$

This exhibits the fallacy of Denying a Conjunct.

g. Let p: "Leona wins the singing competition." and q: "Leona lands a recording contract with a famous company." The argument takes the form

$$\begin{array}{rc} & p \rightarrow q \\ & q \\ \hline \therefore & p \end{array}$$

This is invalid as this is an example of the Fallacy of the converse.

h. Let p: "Michael sleeps early tonight.", q: "Michael plays with his pet dog.", and r: "Michael will wake up early tomorrow." The argument is of the form

$$\begin{array}{rc} & p \rightarrow r \\ & \sim q \rightarrow r \\ \hline \therefore & [p \vee (\sim q)] \rightarrow r \end{array}$$

This is valid due to the Rule of Proof by Cases.

2. Determine whether each of the following arguments is (a) valid and (b) sound.

a. If triangle T_1 and T_2 are congruent, then they are similar. Triangles T_1 and T_2 are congruent. Therefore, triangles T_1 and T_2 are similar.

b. The set of natural numbers is finite or the set of negative integers is finite. It is known that the set of negative integers is infinite. Therefore, the set of natural numbers is finite.

c. Quadrilateral $ABCD$ is a square or a parallelogram. It is known that $ABCD$ is a parallelogram. Therefore, it is not a square.

d. If $\sqrt{2} > \frac{3}{2}$, then $\left(\sqrt{2}\right)^2 > \left(\frac{3}{2}\right)^2$. We know that $\sqrt{2} > \frac{3}{2}$. Therefore, $\left(\sqrt{2}\right)^2 > \left(\frac{3}{2}\right)^2$

e. If n is a real number with $n > 3$, then $n^2 > 9$. Suppose $n^2 \leq 9$. Then $n \leq 3$.

f. If 9 is less than 4, then 9 is not a prime number. 9 is a prime number. Therefore, 9 is not less than 4.

Solution.

a. The argument is valid by Modus Ponens. Furthermore, we know from the geometry of triangles that congruent triangles are also similar (but similar triangles are not necessarily congruent). If it is taken to be true that T_1 and T_2 are congruent, then the argument satisfies the truth condition. Hence, the argument is sound.

b. The argument is valid by Disjunctive Syllogism. However, it is not sound because there are infinitely many natural numbers and negative integers, and this points to the falsity of the first premise.

c. The argument is invalid as it exhibits the fallacy of Affirming the Disjunct, and hence it is unsound.

d. The argument is valid by Modus Ponens. But the premise $\sqrt{2} > \frac{3}{2}$ is not true, so the argument is not sound.

e. The argument is valid by Modus Tollens. The first premise is true by the monotonicity of the function $f(x) = x^2$. If the second premise is taken to be true, then the argument is sound.

f. The argument is valid by Modus Tollens, but the second premise not true; 9 has factors other than 1 and itself. Hence, the argument is not sound.

Lesson 40 Supplementary Exercises

1. Determine whether the following arguments are valid. If it is valid, then identify the rule of inference which justifies its validity. Otherwise, state a counterexample or identify the type of fallacy exhibited by the argument.

a. If x is an odd integer, then $2x + 1$ is also an odd integer. If $2x + 1$ is an odd integer, then $3(2x + 1)$ is an odd integer. Therefore if x is an odd integer, then $3(2x + 1)$ is an odd integer.

b. If u is even and v is odd, then uv is even. It is found that uv is even. Therefore, u is even and v is odd.

c. If quadrilateral $ABCD$ is a square, then it is also a rectangle. Quadrilateral $ABCD$ is not a rectangle. Therefore, it is not a square.

d. If Delight publishes a dissertation with original results, then she will earn a Ph.D. She did not publish a dissertation with original results. Therefore, she did not earn a Ph.D.

e. If Jason has a Ph.D. and has done a considerable amount of research, then he is qualified for the research professor position. He was informed that he is qualified for the research professor position. Therefore, Jason has a Ph.D. and has done a considerable amount of research.

f. If the housing market crashes, then all of my investments will suffer. My investments have not suffered. Therefore, the housing market has not crashed.

g. Either Derrick was not informed about the meeting or he made the decision not to attend. Derrick was not informed about the meeting. Thus, he did not decide not to attend the meeting.

h. It is known that f is a polynomial function and it is a one-to-one function. Therefore, f is a one-to-one function.

2. Determine whether each of the following arguments is (a) valid and (b) sound.

a. If n is a real number with $n > 2$, then $n^2 > 4$. Suppose $n \leq 2$. Then $n^2 \leq 4$.

b. If y is a positive number, then $y^2 > 0$. Suppose $y^2 > 0$. Then y is a positive number.

c. If $\sqrt{2}$ is a rational number, then $\sqrt{2} = a/b$ for some integers a and b. It is not true that $\sqrt{2} = a/b$ for some integers a and b. Therefore, $\sqrt{2}$ is not a rational number.

d. If the polygon is a quadrilateral, then the sum of its interior angles is $360°$. The sum of the interior angles of the polygon is not $360°$. Therefore, the polygon is not a quadrilateral.

e. If at least one of two numbers is divisible by 5, then the product of the two numbers is divisible by 5. Neither of the two numbers is divisible by 5. Therefore, the product of these two numbers is not divisible by 6.

Lesson 41: Methods of Proof

Learning Outcome(s): At the end of the lesson, the learner is able to illustrate different methods of proof.

Lesson Outline:

1. Introduction.
2. Proof and proving validity of arguments in propositional form.
3. Proof and proving validity of arguments in real-life situations.
4. Disproof.
5. Indirect proofs.
6. Proof and proving validity of arguments in mathematics.

Basic Idea of Proofs. The **goal** of the proof is to show that the conclusion logically follows from the given propositions (or premises).

As for the **content** of the proof, each proposition must be a *valid* assertion: they must be based on a given statement (i.e. a premise), or they must follow from the premise via **logical equivalences** or **rules of inferences**.

Example 1. Prove the validity of the following argument:

$$\begin{array}{rc} & p \rightarrow (r \wedge s) \\ & \sim r \\ \hline \therefore & \sim p \end{array}$$

Solution. *Thinking process:* We assume that all propositions over the line are true. From these two propositions, the goal is to establish a logical sequence of propositions to arrive at the conclusion $\sim p$.

A common strategy is to start with the statement *not* involving a conditional (i.e., start with$\sim r$). Now think, if $\sim r$ is true, **how can we reach** $\sim p$?

To do that, we can use Modus Tollens on $p \rightarrow (r \wedge s)$, but first we need to establish that $\sim (r \wedge s)$ is true. Since $\sim r$ is true, then by the Rule of Addition, $(\sim r) \vee (\sim s)$ is true. It follows that $(\sim r) \vee (\sim s) \Leftrightarrow \sim (r \wedge s)$, by De Morgan's Law.

The actual proof is written as follows:

	Proposition	Reason
1	$\sim r$	Premise
2	$(\sim r) \vee (\sim s)$	(1), Rule of Addition
3	$\sim (r \wedge s)$	(2), De Morgan's Law
4	$p \rightarrow (r \wedge s)$	Premise
5	$\sim p$	(3), (4), Modus Tollens

Example 2. Prove the validity of the argument

$$\begin{array}{rc} & (p \wedge r) \rightarrow (\sim q) \\ & (\sim q) \rightarrow r \\ & \sim r \\ \hline \therefore & \sim (p \wedge r) \end{array}$$

Solution. Observe that the Law of Syllogism can be applied to the first two premises: that is, $(p \wedge r) \rightarrow (\sim q)$ and $(\sim q) \rightarrow r$ imply that $(p \wedge r) \rightarrow r$. This is a new proposition that we can assume to be true.

Also, since $\sim r$ is true, then $\sim (p \wedge r)$ is true by Modus Tollens. The actual proof is written below.

	Proposition	Reason
1	$(p \wedge r) \rightarrow (\sim q)$	Premise
2	$(\sim q) \rightarrow r$	Premise
3	$(p \wedge r) \rightarrow r$	(1), (2), Law of Syllogism
4	$\sim r$	Premise
5	$\sim (p \wedge r)$	(3), (4), Modus Tollens

An alternative proof goes without starting from the conditional premise.

	Proposition	Reason
1	$\sim r$	Premise
2	$(\sim q) \rightarrow r$	Premise
3	$\sim (\sim q)$	(1), (2), Modus Tollens
4	$(p \wedge r) \rightarrow (\sim q)$	Premise
5	$\sim (p \wedge q)$	(3), (4), Modus Tollens

Example 3. Prove the validity of the following argument.

$$\begin{array}{rc} & p \vee r \\ & (\sim r) \vee (\sim s) \\ & s \\ \hline \therefore & p \end{array}$$

Solution. We can start with the simple proposition s. Then $\sim s$ must be false since s is taken to be true. By Disjunctive Syllogism $(\sim r) \vee (\sim s)$, it follows that $\sim r$ is true. Applying Disjunctive Syllogism again with $p \vee r$, it follows then that p is true.

The actual proof is written below.

	Proposition	Reason
1	s	Premise
2	$\sim (\sim s)$	Double Negation
3	$(\sim r) \vee (\sim s)$	Premise
4	$\sim r$	(2), (3), Disjunctive Syllogism
5	$p \vee r$	Premise
6	p	(4), (5), Disjunctive Syllogism

Alternatively, it is also valid to transform the premises $p \vee r$ and $(\sim r) \vee (\sim s)$ to $\sim p \rightarrow r$ and $r \rightarrow \sim s$, respectively, using the Switcheroo Law. Then we can use Modus Tollens and the Law of Syllogism, as shown below:

	Proposition	Reason
1	$p \vee r$	Premise
2	$\sim p \rightarrow r$	Switcheroo
3	$(\sim r) \vee (\sim s)$	Premise
4	$r \rightarrow \sim s$	Switcheroo
5	$\sim p \rightarrow \sim s$	Law of Syllogism
6	s	Premise
7	$\sim (\sim s)$	(6), Double Negation
8	$\sim (\sim p)$	(5), (7), Modus Tollens
9	p	Double Negation

Example 4. Analyze the validity of the following argument:
If you start your own business, then you will earn right away.
If you go to college, then you will get a college degree after a few years.
However, you either start your own business, or you go to college.
Therefore, either you earn right away, or get a college degree after a few years.

Solution. We transform the given argument in symbols. Define the propositions b: "You start your own business.", e: "You earn right away.", c: "You go to college.", and d: "You get a college degree after a few years."
In standard form, the argument is

$$\begin{array}{rc} & b \rightarrow e \\ & c \rightarrow d \\ & b \vee c \\ \hline \therefore & e \vee d \end{array}$$

This is a valid argument, as shown in the following proof:

	Proposition	Reason
1	$b \vee c$	Premise
2	$\sim (\sim b) \vee c$	Double Negation
3	$(\sim b) \rightarrow c$	Switcheroo
4	$c \rightarrow d$	Premise
5	$(\sim b) \rightarrow d$	(3), (4), Law of Syllogism
6	$b \rightarrow e$	Premise
7	$(\sim e) \rightarrow (\sim b)$	(4), The contrapositive is logically equivalent to the original conditional
8	$(\sim e) \rightarrow d$	(5), (7), Law of Syllogism
9	$\sim (\sim e) \vee d$	Switcheroo
10	$e \vee d$	Double Negation

Note: We have shown in the previous example that an argument of the form

$$\begin{array}{rl} & p \to q \\ & r \to s \\ & \underline{p \vee r} \\ \therefore & q \vee s \end{array}$$

is valid. This form is called the **constructed dilemma.**

Example 5. Show that the following argument is invalid: "I would like a career in either teaching or diplomacy. If I teach, then I would want to study abroad. Therefore, if I would like a career in diplomacy, then I will study abroad."

Solution. We first write the argument in symbolic form using the following propositions:

t: I would like a career in teaching.

d: I would like a career in diplomacy.

s: I would want to study abroad.

Thus, the argument can be written in standard form as

$$\begin{array}{rl} & t \vee d \\ & \underline{t \to s} \\ \therefore & d \to s \end{array}$$

To show that an argument is not valid, we need to find truth values for each proposition such that the premises are true, but the conclusion is false.

For $d \to s$ to be false, then d must be true and s must be false. If t is false, then $t \vee d$ is true and $t \to s$ are both true. Since there is such a combination of truth values for t, s, and d that makes the conclusion false but the premises true, the argument is invalid.

This is an example illustrating that **producing a counterexample** is sufficient to show that an argument is invalid.

Another method is through the use of an **indirect proof** or a **proof by contradiction**. In these proofs, **we show that the assumption that the premises are true but the conclusion is false leads to a contradiction.**

Example 6. Prove the following argument using three methods: (a) via the rules of inference, (b) via truth tables, and (c) via an indirect proof.

$$\begin{array}{rl} & p \vee q \\ & \underline{\sim q} \\ \therefore & p \end{array}$$

Solution.

a. Via rules of inference

	Proposition	**Reason**
1	$p \vee q$	Premise
2	$\sim q$	Premise
3	p	(1), (2), Disjunctive Syllogism

b. Via truth tables

p	q	$p \vee q$	$\sim q$	$[(p \vee q) \wedge (\sim q)]$	$[(p \vee q) \wedge (\sim q)] \rightarrow p$
T	T	T	F	F	T
T	F	T	T	T	T
F	T	T	F	F	T
F	F	F	T	F	T

Since $[(p \vee q) \wedge (\sim q)] \rightarrow p$ is a tautology, the argument is valid.

c. Via indirect proof

We assume the conclusion is false, while the premises are true, and show that these lead to a contradiction.

Suppose p (the conclusion) is false. Based on the premise, $\sim q$ is true, and so q is false. Therefore, $p \vee q$ is false, which is a contradiction of the premise that $p \vee q$ is true.

Therefore, the conclusion must be true.

We now apply the rules of logic to prove basic results in mathematics. Before we proceed, we state the following important idea:

Definitions in mathematics are always "if and only if" statements.

Consider the following examples.

Definition	"If and only if" Form
An even number m is a number than can be written as $m = 2k$, where k is an integer	A number m is even if and only if it can be written as $m = 2k$, where k is an integer.
A binomial is a polynomial with exactly 2 terms.	A polynomial is a binomial if and only if it has exactly 2 terms.
A parallelogram is a quadrilateral with two pairs of parallel sides.	A quadrilateral is a parallelogram if and only if it has two pairs of parallel sides.

Example 7. Prove the validity of the following argument.

An even number m is a number than can be written as $m = 2k$, where k is an integer. The numbers x and y are even. Therefore, $x + y$ is even.

Solution. Since x is even, then the first premise ensures that x can be written as $x = 2k_1$, where k_1 is an integer. Similarly, since y is even, then we can write $y = 2k_2$, where k_2 is an integer.

We compute for $x + y$:

$$x + y = 2k_1 + 2k_2 = 2(k_1 + k_2).$$

Since $k_1 + k_2$ is an integer, then $x + y$ is even. This is the conclusion when Modus Ponens is applied to the first premise.

Example 8. Prove that the following argument is valid.

If a quadrilateral has three right angles, then it is a rectangle. In quadrilateral $ABCD$, $m\angle A = 90°$, $m\angle B = 90°$, and $m\angle C = 85°$. Then, $ABCD$ is not a rectangle.

Solution.

Thinking process: The only way for $ABCD$ to be a rectangle is if $m\angle D = 90°$, so that there would be three triangles. We will prove that this is *not* the case.

Proof: The sum of the interior angles in a rectangle is $360°$. Therefore,

$$m\angle A + m\angle B + m\angle C + m\angle D = 360°.$$

Substituting the values given, we have $90° + 90° + 85° + m\angle D = 360°$, which simplifies to $m\angle D = 95°$. Therefore, $ABCD$ has only two right angles. This also means that $ABCD$ is not a rectangle (using Modus Tollens on the definition stated in the first premise).

Indirect Proof: Assume that $ABCD$ is a rectangle. Then it has three right angles. But since $\angle C$ is not a right angle, then the three right angles must be $\angle A$, $\angle B$, and $\angle D$.

Solving for $m\angle C$ in the equation $m\angle A + m\angle B + m\angle C + m\angle D = 360°$ (given $m\angle A = 90°$, $m\angle B = 90°$, and $m\angle D = 90°$), we find that $m\angle C = 90°$. This contradicts the fact that $m\angle C = 85°$, which is the given. Therefore, $ABCD$ is not a rectangle.

Solved Examples

Determine whether the following arguments are valid using rules of inference. If the argument is invalid, provide a counterexample.

1. An odd integer m is a number that can be written in the form $m = 2k + 1$, where k is an integer. It is known that x and y are odd integers. Therefore, $x + y$ is even.

2. Mary is studying computing or Mary is not studying math. If Mary is studying math, then Mary is not studying computing. Therefore, Mary is studying computing.

3. An odd integer m is a number that can be written in the form $m = 2k + 1$, where k is an integer. We know that n is odd. Therefore, n^2 is odd.

4. If the machine is cheap or is energy efficient, then it will not make money for the manufacturer. If the machine is painted red, then it will make money for the manufacturer. The machine is cheap. Therefore, it is not painted red.

5. If monsters roam the Earth, then all people will buy weapons. If Earth experiences tremors from beneath the surface, then all people will evacuate. Monsters roam the Earth and people are evacuating. Therefore, all people bought weapons, and the Earth experienced tremors from beneath the surface. (Invalid: A true, B true, C false, D true)

6. If Cherry's song is loud or tedious, then it is not long and not cacophonous. Cherry's song is tedious. Therefore, Cherry's song is not long.

7.

$$\begin{array}{rc} & p \vee q \\ & \sim p \vee r \\ \hline \therefore & q \vee r \end{array}$$

8.

$$\begin{array}{rc} & d \rightarrow f \\ & s \rightarrow (\sim f) \\ & d \wedge s \\ \hline \therefore & p \end{array}$$

9.

$$\begin{array}{rc} & r \rightarrow j \\ & j \rightarrow (\sim f) \\ & v \rightarrow f \\ & r \vee v \\ \hline \therefore & j \vee f \end{array}$$

10.

$$\begin{array}{rc} & \sim (a \vee c) \\ & (\sim c) \vee u \\ & u \rightarrow s \\ \hline \therefore & s \end{array}$$

Solution.

1. The first premise is a definition, and can thus be interpreted as an if-and-only-if statement. Suppose x and y are odd. Then we can write $x = 2k_1 + 1$ and $y = 2k_2 + 1$ for some integers k_1 and k_2. Thus,

$$x + y = 2k_1 + 1 + 2k_2 + 1 = 2(k_1 + k_2 + 1).$$

Since $k_1 + k_2 + 1$ is an integer, then by Modus Ponens on the definition of even numbers, $x + y$ is even.

2. Let c: "Mary is studying computing." and m: "Mary is studying math."

In propositional form, the argument is a: $\{[c \vee (\sim m)] \wedge [m \to (\sim c)]\} \to c$. Validity can be checked with a truth table

c	m	$\sim c$	$\sim m$	$c \vee (\sim m)$	$m \to (\sim c)$	$[c \vee (\sim m)] \wedge [m \to (\sim c)]$	a
T	T	F	F	T	F	F	T
T	F	F	T	T	T	T	T
F	T	T	F	F	T	F	T
F	F	T	T	T	T	T	F

The argument is not valid since a is not a tautology. A counterexample is when c and m are both false.

3. Suppose n is odd. Then we can write $n = 2k + 1$ for some integer k. Furthermore,

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Since k is an integer, then $2k^2 + 2k$ is also an integer. Then by Modus Ponens, n^2 is also odd.

4. Let c: "The machine is cheap.", e: "The machine is energy efficient.", m: "The machine will make money for the manufacturer.", and r: "The machine is painted red." In standard form, the argument is

$$\begin{array}{rc} & (c \vee e) \to (\sim m) \\ & r \to m \\ & c \\ \hline \therefore & \sim r \end{array}$$

Proof:

	Proposition	Reason
1	$r \to m$	Premise
2	$(\sim m) \to (\sim r)$	(1), Equivalence to contrapositive
3	c	Premise
4	$c \vee e$	(3), Addition Law
5	$(c \vee e) \to (\sim m)$	Premise
6	$(c \vee e) \to (\sim r)$	(2), (5), Law of Syllogism
7	$\sim r$	(4), (6), Modus Ponens

5. Suppose m: “Monsters roam the Earth.”, w: ”All people will buy weapons.”, t: “Earth experiences tremors from beneath the surface.”, and e: “All people will evacuate.” Constructing a truth table shows that the case m, w, and e are true and t is false is a counterexample. The argument is invalid.

6. Let l: “Cherry’s song is loud.”, t: “Cherry’s song is tedious.”, L: “Cherry’s song is long.”, and c: “Cherry’s song is not cacophonous.” The argument, in standard form is

$$\begin{array}{rc} & (l \vee t) \rightarrow [(\sim L) \wedge (\sim c)] \\ & t \\ \hline \therefore & \sim L \end{array}$$

Proof:

	Proposition	Reason
1	t	Premise
2	$l \vee t$	Addition Law
3	$(l \vee t) \rightarrow [(\sim L) \wedge (\sim c)]$	Premise
4	$(\sim L) \wedge (\sim c)$	(2), (3), Modus Ponens
5	$\sim L$	Simplification Law

7.

	Proposition	Reason
1	$(\sim p) \vee r$	Premise
2	$p \rightarrow r$	(1), Switcheroo
3	$p \vee q$	Premise
4	$\sim (\sim q) \vee p$	(3), Double Negation, Commutative Law
5	$\sim q \rightarrow p$	(4), Switcheroo
6	$\sim q \rightarrow r$	(2), (5), Law of Syllogism
7	$q \vee r$	(6), Switcheroo, Double Negation

8.

	Proposition	Reason
1	$d \rightarrow f$	Premise
2	$s \rightarrow (\sim f)$	Premise
3	$d \wedge s$	Premise
4	d	(3), Simplification
5	s	(3), Simplification
6	f	(1), (4), Law of Syllogism
7	$\sim f$	(2), (5), Law of Syllogism
8	$f \wedge (\sim f)$	(6), (7), Law of Conjunction
9	ϕ	(8), Inverse Law
10	$\phi \vee p$	(9), Addition Law
11	p	(10), Identity Law

9.

	Proposition	Reason
1	$r \to j$	Premise
2	$j \to (\sim f)$	Premise
3	$r \to (\sim f)$	(1), (2), Law of Syllogism
4	$v \to f$	Premise
5	$r \vee v$	Premise
6	$(\sim f) \vee f$	(3), (4), (5), Constructed Dilemma
7	τ	(6), Inverse Law
8	$\tau \vee (j \vee f)$	(7), Addition Law
9	$j \vee f$	(8), Identity Law

10. This is invalid. A counterexample is when all propositions involved are false.

Prove each of the following statements.

1. If n is even and m is odd, then $n + m$ is odd.

2. Suppose x and y are divisible by 3. Then $x + y$ is also divisible by 3.

3. If n^2 is odd, then n is odd.

Solution.

1. Suppose n is even and m is odd. Then for some integers k_1 and k_2, we can write $n = 2k_1$ and $m = 2k_2 + 1$. Therefore,

$$n + m = 2k_1 + 2k_2 + 1 = 2(k_1 + k_2) + 1.$$

Since $k_1 + k_2$ is also an integer, then $n + m$ is odd (as it has the form of an odd number).

2. Suppose x and y are divisible by 3. Then for some integers k_1 and k_2, we can write $x = 3k_1$ and $y = 3k_2$. Therefore,

$$x + y = 3k_1 + 3k_2 = 3(k_1 + k_2).$$

Since $k_1 + k_2$ is an integer, then $x + y$ is also divisible by 3.

3. (By contradiction) Suppose the conclusion "n is odd" is false; that is, suppose n is even. Therefore, we can write $n = 2k$ for some integer k. This means that

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

We note that k^2 is also an integer, implying that n^2 is even. This contradicts the premise that n^2 is odd. Therefore, n must be odd.

Lesson 41 Supplementary Exercises

A. Determine whether the following arguments are valid using rules of inference. If the argument is invalid, provide a counterexample.

1. An odd integer m is a number that can be written in the form $m = 2k + 1$, where k is an integer. It is known that x and y are odd integers. Therefore, $x + 2y$ is odd.

2. Noel is not cooking or Noel is watching a television show. If Noel is not watching a television show, then Noel is cooking. Therefore, Noel is not cooking.

3. An even number m is a number than can be written as $m = 2k$, where k is an integer. It is given that y is an even number. Therefore, y^2 is also even.

4. If Michelle wakes up early or does not sleep, then she will get to class on time. If she commutes to school, then she will not get to class on time. Michelle did not sleep. Therefore, she did not commute to school.

5. If lightning strikes a tree on the farm, then all the animals will make sounds of horror. If thunder booms in the sky, then all the animals will run back to the stables. Lightning struck a tree and all the animals ran back to the stables. Therefore, all animals made sounds of horror and thunder boomed in the sky.

6. It is not the case that Joe plays both piano and violin. If Joe does not play piano and he does not play violin, then he plays both clarinet and drums. If he plays drums, then he plays the clarinet. Therefore, Fred plays clarinet.

7.

$$\begin{array}{rc} & p \wedge q \\ & (p \vee q) \rightarrow r \\ \hline \therefore & r \end{array}$$

8.

$$\begin{array}{rc} & s \rightarrow i \\ & (\sim i) \vee n \\ & \sim (p \vee s) \\ \hline \therefore & n \end{array}$$

9.

$$\begin{array}{rc} & f \rightarrow o \\ & (\sim g) \rightarrow (\sim o) \\ & g \rightarrow v \\ & \sim v \\ \hline \therefore & \sim f \end{array}$$

10.

$$\begin{array}{rc} & s \to e \\ & s \vee c \\ & \underline{\sim (e \vee c)} \\ \therefore & m \end{array}$$

B. Prove each of the following statements.

1. If n is odd and m is even, then $2n + 3m$ is even.

2. If x and y are divisible by 4, then xy is also divisible by 4.

3. If n^2 is even, then n is even.

Topic Test 1

1. Prove that $p \wedge q$ is logically equivalent to $\sim (p \to (\sim q))$ using

a. the known logical equivalences

b. a truth table

2. Determine whether the following argument is valid using rules of inferences. If it is invalid, give a counterexample

Jim works at a paper company, but he does not get paper cuts. Either Jim gets a paper cut, or he makes deliveries to his clients. Therefore, if he does not make deliveries to his clients, he works at a paper company.

3. Determine whether the following argument is sound. Otherwise, explain why it is not sound.

If the equation $x^2 + 2 = 0$ has a real solution, then the solution set of the inequality $x + 3 \leq 4$ is the interval $(-\infty, 1]$. If 59 is divisible by 7, then the solution set of the inequality $x + 3 \leq 4$ is not the interval $(-\infty, 1]$. Therefore, if the equation $x^2 + 2 = 0$ has a real solution, then 59 is not divisible by 7.

4. Prove or disprove: the sum of any three consecutive integers is divisible by 3.

Topic Test 2

1. Write the following argument in symbols:

If the pot is hot, then Caleb cannot carry it. If Caleb was able to carry the pot, then he can serve it on the table. Caleb was able to carry the pot. Therefore, the pot is not hot and he can serve it on the table.

2. Prove that the proposition

"If Jude wears a black shirt, then if he wears jeans then he will look like me."

and the proposition

"If Jude wears a black shirt and jeans, then he will look like me."

are logically equivalent using known logical equivalences.

3. Determine whether the following argument is valid using rules of inference. Otherwise, provide a counterexample to show that the argument is invalid.

$$\begin{array}{rc} & (p \wedge q) \rightarrow (\sim r) \\ & p \vee (\sim q) \\ & \underline{(\sim q) \rightarrow p} \\ \therefore & \sim r \end{array}$$

4. Prove: If n is an integer, then $3n + 5$ is not divisible by 3. (Hint: Argue using a contradiction).

Lesson 1

1. k = 2,4

3. V and W

5. 5. S(n) = 600n

7.
$$R(t) = \begin{cases} 20\lceil t \rceil & if\ 0 < t \leq 2 \\ 40 + 10\lceil (t-2) \rceil & if\ t > 2 \end{cases}$$

Lesson 2

1.

(a) -128
(b) 125
(c) 3
(d) -4 is not in the domain of $q(x)$

3.

(a) 1
(b) $x^2 + 2x + 1$

$f(x) + f(3) = x^2 + 4x + 5 \neq f(x+3)$

5. N(5) ≈ 0.17 and N(15) ≈ 0.65

Lesson 3

1.

(a) $(f+g)(x) = x^2 + x - 6$
(b) $(f-g)(x) = -x^2 + x - 4$
(c) $(f \cdot g)(x) = x^3 - 5x^2 - x + 5$
(d) $\left(\frac{f}{g}\right)(x) = \frac{x-5}{x^2-1}$
(e) $\left(\frac{g}{f}\right)(x) = \frac{x^2-1}{x-5}$

3.

(e) $(fog)(x) = \frac{1}{x^2} - 1$
(f) $(gof)(-1) = \frac{1}{2}$
(g) $(fof)(x) = x^4 - 2x^2$
(h) $(gog)(5) = 5$

5. (a) 320x – 8x^2 (0 ≤ x ≤ 10); gross income for selling x bags

(b) 120x – 8x^2 (0 ≤ x ≤ 10); profit or net income from selling x bags

Lesson 4

1.

(a) 3
(b) -1
(c) 12
(d) 3

3.

(a) $f(x) = \frac{90\,000}{x+1}$
(b) $f(5) = \frac{90\,000}{5+1} = 15,000$

The marketing office gets twice that amount which is P30,000 while the other offices receive P15,000 each.

Lesson 5

1. Rational function

3. None of these

5. None of these

Lesson 6

1. $-16/3$

3.The solution set is

$\{x \in \mathbb{R} | \ x < -3 \ or \ x \geq 1\}$.

5. The solution set is given by $\{x \in \mathbb{R} | -5 \leq x < 1 \ or \ x \geq 6\}$.

7. 6 liters

Lesson 7

1.

(a)

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-1	-1.2	-1.5	-2	-3	-6	und	6	3

(b)

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-1	-1.2	und	-2	-3	-6	und	6	3

3.

(a)

t	1	10	100	1000	10000	100000
$f(t)$	182	550	918	991	999	1000

(b)

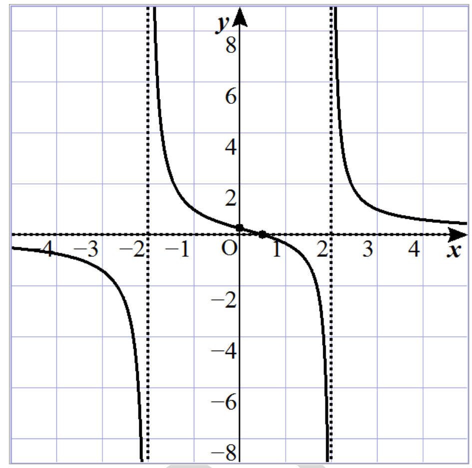
The lake seems to be able to support only 1000 fishes.

Lesson 8

1. Horizontal asymptote: $y = 0$

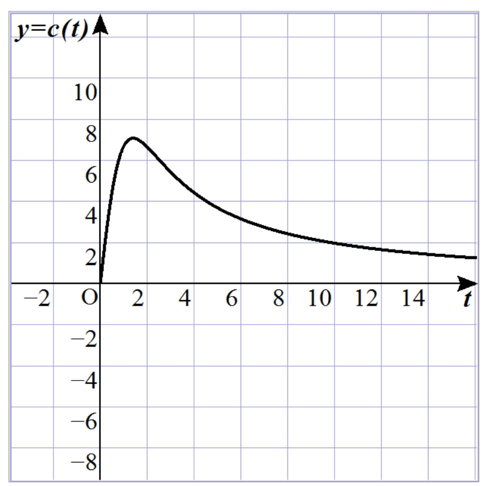
Vertical asymptotes: $x = 2, x = 6$

3. The x-intercept is $\frac{1}{2}$. The y-intercept is $\frac{1}{4}$. The vertical asymptotes are $x = -2$ and $x = 2$. The horizontal asymptote is y = 0. The domain is $\{x \in \mathbb{R} | x \neq \pm 2\}$. The function is above the x-axis in the intervals –2 < x < ½ and x > 2. The function is below the x-axis in the intervals x < –2 and ½ < x < 2. The graph is shown below. The range is $\{f(x) \in \mathbb{R} | f(x) \neq 0\}$.



5. The graph is shown at the right (calculus is needed to determine the maximum height. At the level of Grade 11, it suffices to check that the graph approaches the horizontal asymptote, and is above the x-axis.)

The horizontal asymptote is y = 0. In the long run, the concentration in the bloodstream decreases and becomes almost negligible.



Lesson 9

1. $k = 2, 4,$ or 6

3. Yes, the function is one-to-one

Lesson 10

1.

(a) Has an inverse
(b) Has an inverse
(c) Has no inverse
(d) Has no inverse
(e) Has no inverse

3. $f(x) = \frac{1}{x} + 2$

Lesson 11

1. Domain: $\{x \in \mathbb{R} | -4 < x < 5\}$.

Range: $\{f(x) \in \mathbb{R} | \ 0 < f(x) < 3\}$.

3. $(5,5)$

5. The function with restricted domain L ≥ 0 is one-to-one. The inverse function is $L = \sqrt{\frac{w}{3.24 \times 10^{-3}}}$. If w = 0.4, then L = 100/9 ≈ 11.1 cm.

Lesson 12

1. y = $1{,}000(2)^{t/60}$; 1122

3. y = $300(1/2)^{t/1200}$; 168 g

Lesson 13

1. Exponential Equation

3. Transformation of an Exponential Function

5. Exponential Inequality

Lesson 14

1. x = 0

3. x = –3

5. $\{x \in \mathbb{R} | \ x \leq -3/2\}$

7. x = –8

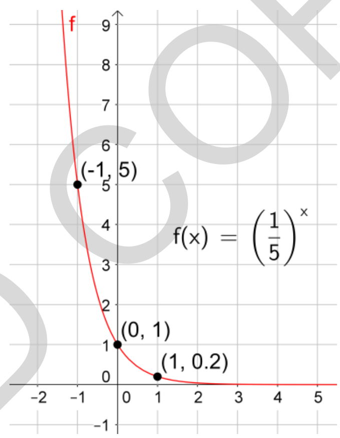
9. $\{x \in \mathbb{R} | \ x > -2\}$

11. 14.536 days

Lesson 15

1.

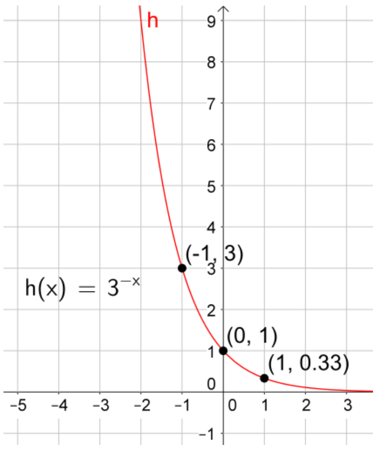
x	–2	–1	0	1	2
f(x)	25	5	1	1/5	1/25



Domain: All real numbers; Range: All positive real numbers; y-intercept: 1; horizontal asymptote: y = 0

3.

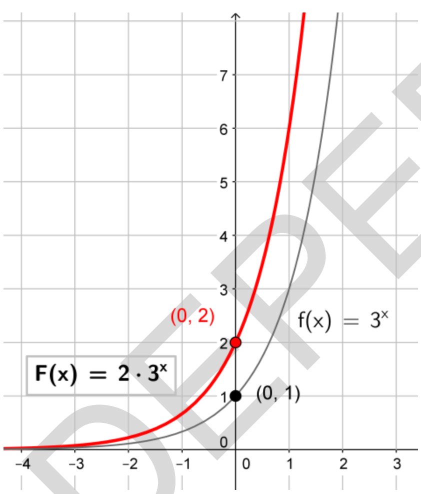
x	–2	–1	0	1	2
f(x)	9	3	1	1/3	1/9



Domain: All real numbers; Range: All positive real numbers; y-intercept: 1; horizontal asymptote: y = 0

Lesson 16

1. Base function: $f(x) = 3^x$



Transformation/s: Stretch the graph of f(x) by 2 units.

Domain: All real numbers

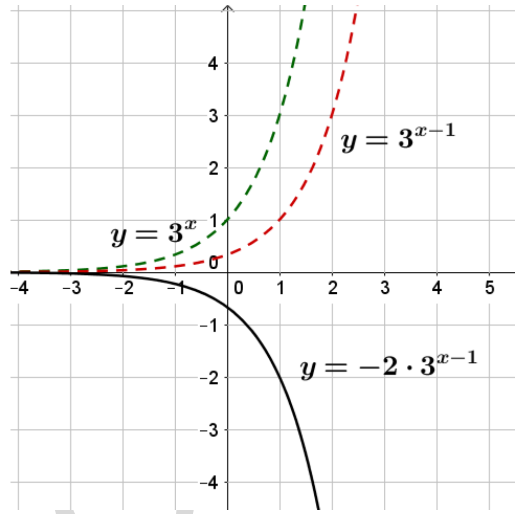
Range: $(0,\infty)$;

y-intercept: (0,2)

Horizontal Asymptote: y = 0

3. Base function: $h(x) = 3^x$

Transformations: Shift the graph of h(x) 1 unit to the right, reflect about the x-axis, then stretch 2 units.



Domain: All real numbers
Range: $(-\infty, 0)$
y-intercept: (0, –2/3)
Horizontal Asymptote: y = 0

5. $[3^{-10}, 3^{10}]$

Lesson 17

1. –1

3. –2

5. $\log_{144} 12 = \frac{1}{2}$

7. $2^3 = 8$

9. $5^{-2} = \frac{1}{25}$

11. basic

Lesson 18

1. Logarithmic Inequality

3. Logarithmic Function

5. Logarithmic Equation

Lesson 19

1. –2

3. 0

5. 3

Lesson 20

1. $4\log_b x + \frac{1}{3}\log_b y$

3. $\frac{1}{2}\log x + \frac{1}{3}\log y$

5. $\ln\left[\left(\sqrt{x+3}\right)\left(\sqrt[3]{x+2}\right)\right]$

7. $\frac{\log 1/2}{\log 5} \approx -0.4307$

9. $\frac{\ln 2.5}{\ln 7.2} \approx 0.4642$

Lesson 21

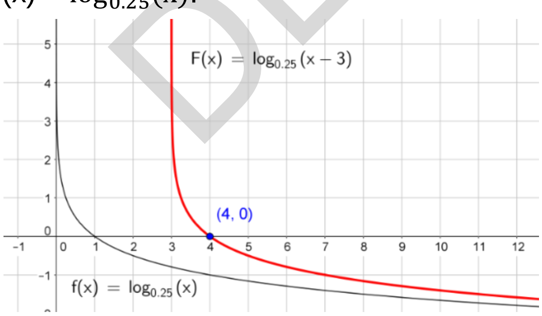
1. x = 12

3. x = 4 or x = ¼

7. $15e^{-0.9} \approx 6.10$ lumens

Lesson 22

1. The graph of F(x) is a shift of 3 units to the right from the graph of f(x) = $\log_{0.25}(x)$.



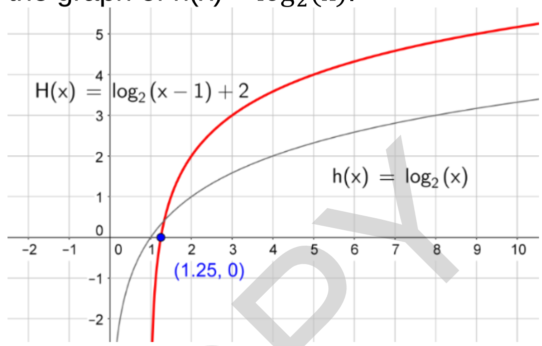
Domain: { x ∈ ℝ | x > 3 }

Range: all real numbers

Vertical Asymptote: x = 3

x-intercept: (4, 0)

3. The graph of H(x) is a shift of 1 unit to the right and 2 units upward from the graph of h(x) = $\log_2(x)$.



Domain: { x ∈ ℝ | x > 1 }

Range: all real numbers

Vertical Asymptote: x = 1

x-intercept: (1.25, 0)

Lesson 24

A.

1. 300

3. 2.27%

5. 60,000

B.

7. 11.840
9. 1,000

C.
11. P 39,000

13. P 33,750

15. 5 years

17. 8.33%

19. P 90.909.09

Lesson 25

A.

1. 9,109.02

3. 5,224.05

5. 4,836.85

7. 23,551.13

9. 288,882.945

B.

11. P 116,640

13. P 141,856.71

15. P 78,849.32

17. P 36,787.10

19. P 122,657.58

Lesson 26

1. 2

3. 6% or 0.06

5. 1% or 0.01

7. 16

9. 32,094.13

11. 10% or 0.1

13. P 52,186.20

15. P149,763.81

17. (a) 281,789.94

(b) 512,687.70

(c) 230,898.26

19. Bank A: 216,756.59

Bank B: 212,643.79

Lesson 27

A.

1. 4

3. 0.10125 or 10.125%

5. 0.03 0r 3 %

7. 0.004939 or 0.4939%

9. 0.121204 or 12.1204%

B.

11. 4

13. 0.003979 or 0.398%

15. 800

17. 0.055 or 5.5%

19. 2.34 periods

C.

21. j = 0.0301 or 3.01%

$i^{(2)}$ = 0.0602 or 6.02%

23. n = 11.485 periods

t = 0.97 year or 11 ½ months

25. r_s = 0.02015 or 2.015%

Lesson 28

A.

1. 152,793.63

3. 201,867.57

5. 661.78

B.

7. 410,332.19

9. 1,006,512.21

C.

11. 3,979.13
13. 28,859.15

15. 3,391.91

D.
17. P 459,803.80

19. P 22,456.46

Lesson 29

A.
1. P152,770.90
3. P98,352.04
5. P 604.37

B.

7. P 414,489.40

9. P 2,052,510

C.

11. P 3,980.64

13. P 59,449.85

15. P3,448.11

D.

17. P 857,260.30

19. Sunrise Investment: F = P 109,755.10 (larger)

XYZ Investment: F = P 106,542.70

Lesson 30

A.

1. k=8

3. k=3

5. k=7

7.k=6

9. k=2

B.

1. P=154,694.04
3. P=902,667.55
5. P=3,021,151.03
7. P=152,201.66
9. P=24,112.74

Lesson 31
1. Dividend=20,250
3. Dividend per Share = 46

5. Stock yield ratio = 0.65

7. P2,475

9. P518,526.46

Lesson 32

1. Php 75

3. 2.8%

5. Php 68

7. 12,200

9. Php 28

Lesson 33

1. True 3. False

5. True 7. True

Lesson 34

1. Business Loan

3. Consumer Loan

5. Consumer Loan

Lesson 35

1. P1,225,043
3. P59,557.12
5. P960,000
7. P1,960,000
9. P2,700,000
11. P2,538,000
13. P4,573.65
15. P13,290.89
17. 156
19. P466,841.46
21. P5,795.48
23. P326.12
25. P2,984.70
27. A. Php100,00

    B. Php31,547.08

    C. P7,845.29

    D. P23,701.79

    E. 0

Lesson 36

1. This is not a proposition.

3. This is a compound proposition of the form **if** n_1 **then** n_2, where n_1: “Ted’s score is less than 50” and n_2: “Ted will fail the course”.

5. This is a compound proposition of the form p_1 **or** p_2, where p_1: “It is sunny in Manila” and p_2: ”Its streets are flooded.”

7. This is a compound proposition of the form **if** r_1 **then** r_2, where r_1: “a, b, and c denote the lengths of the legs and the hypotenuse of a right triangle” and n_2: “$a^2 + b^2 = c^2$”.

9. This is a compound proposition of the form **not** t_1, where t_1: “-5 is a negative number”.

11. This is a compound proposition of the form **if** v_1 **then** v_2, where v_1: “If Jerry receives a scholarship” and n_2: “he will go to college”.

13. This is a compound proposition of the form **if [(not** x_1) **or (not** x_2)**], then (not** x_3), where x_1: “You run 1 kilometer a day”, x_2: “You eat property”, and x_3: “You will be healthy.”

Lesson 37

1.

$\sim m$: Mathematics is not easy to study.

$\sim o$: Logic is taken up in junior high school.

$\sim q$: There are more than three people in the meeting room at the moment.

2.

a. $q \wedge (\sim r)$;   c. $p \rightarrow \left(q \leftrightarrow (\sim r)\right)$

e. "If Rena eats at the Spanish restaurant, then if she does not order the restaurant's special paella then she has dessert."

3.

a. $w \wedge (u \wedge (\sim v))$;   c. $w \leftrightarrow (u \vee v)$

e. "Pam works as a sales associate if and only if Bryan does not have a temporary position in the office, and if Bryan does not have a temporary position in the office, then Pam and Bryan are co-workers."

4.

a. False ;   c. False;   e. True

5.

a. p and q must be false.

Lesson 38

1.

a.

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge [\sim (p \wedge q)]$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

c. Let r be the proposition $(p \leftrightarrow q) \wedge ((\sim p) \rightarrow q)$

p	q	$\sim p$	$p \leftrightarrow q$	$(\sim p) \rightarrow q$	r
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	F	T	F
F	F	T	T	F	F

e. Let s denote $(p \to (q \to r)) \to \big((p \wedge q) \to r\big)$

p	q	r	$q \to r$	$p \to (q \to r)$	$p \wedge q$	$(p \wedge q) \to r$	s
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	F	T	T
F	T	F	F	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

This shows that $(p \to (q \to r)) \to \big((p \wedge q) \to r\big)$ is a tautology.

2.

a.

p	q	$\sim p$	$(\sim p) \vee q$	$p \to q$	$\big((\sim p) \vee q\big) \to (p \to q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Therefore, $\big((\sim p) \vee q\big) \to (p \to q)$ is a tautology.

c. Let r denote the proposition $[\sim (p \leftrightarrow q)] \leftrightarrow \big[(p \vee q) \wedge \big(\sim (p \wedge q)\big)\big]$

p	q	$p \leftrightarrow q$	$\sim (p \leftrightarrow q)$	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \big(\sim (p \wedge q)\big)$	s
T	T	T	F	T	T	F	F	T
T	F	F	T	T	F	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	F	F	F	T	F	T

Hence, $[\sim (p \leftrightarrow q)] \leftrightarrow \big[(p \vee q) \wedge \big(\sim (p \wedge q)\big)\big]$ is a tautology.

e. Let s denote the proposition $[p \vee (q \wedge r)] \to [(p \vee q) \wedge (p \vee r)]$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$	s
T	T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F	T
F	F	T	F	F	F	T	F	T
F	F	F	F	F	F	F	F	T

Hence $[p \vee (q \wedge r)] \to [(p \vee q) \wedge (p \vee r)]$ is a tautology.

Lesson 39

1.

a. The given proposition is of the form $p \rightarrow q$, where p: "The clothes are neatly stacked and pressed" and q: "The house help arrived today".

Converse $q \rightarrow p$	"If the house help arrived today, then the clothes are neatly stacked and pressed."
Contrapositive $(\sim q) \rightarrow (\sim p)$	"If the house help did not arrive today, then the clothes are not neatly stacked and pressed."
Inverse $(\sim p) \rightarrow (\sim q)$	"If the clothes are not neatly stacked and pressed, then the house help did not arrive today."

c. We can rewrite the given conditional as "If Nico studied alone, then he got the highest score in the class". This is now of the form $p \rightarrow q$, where p: "Nico studied alone" and q: "He got the highest score in the class".

Converse $q \rightarrow p$	"If he got the highest score in the class, then Nico studied alone."
Contrapositive $(\sim q) \rightarrow (\sim p)$	"If he did not get the highest score in the class, then Nico did not study alone."
Inverse $(\sim p) \rightarrow (\sim q)$	"If Nico did not study alone, then he did not get the highest score in class."

e. The given proposition is of the form $p \rightarrow q$, where p: "Mariah hit the high whistle note" and q: "The audience gave her a standing ovation".

Converse $q \rightarrow p$	"If the audience gave her a standing ovation, then Mariah hit the high whistle note."
Contrapositive $(\sim q) \rightarrow (\sim p)$	"If the audience did not give her a standing ovation, then Mariah did not hit the high whistle note."
Inverse $(\sim p) \rightarrow (\sim q)$	"If Mariah did not hit the high whistle note, then the audience did not give her a standing ovation."

2. The solutions presented below make use of known logical equivalences. Proofs via truth tables will be left as an exercise to the student.

a. We can start with the right-hand side of the equivalence.

		Reason
	$\sim [(\sim p) \wedge (\sim q)]$	Given
$\Leftrightarrow$	$[\sim (\sim p)] \vee [\sim (\sim q)]$	De Morgan's Law
$\Leftrightarrow$	$p \vee q$	Double Negation

c. We can start with the right-hand side of the equivalence.

		Reason
	$\sim [p \wedge (\sim q)]$	Given
$\Leftrightarrow$	$(\sim p) \vee [\sim (\sim q)]$	De Morgan's Law
$\Leftrightarrow$	$(\sim p) \vee q$	Double Negation
$\Leftrightarrow$	$p \rightarrow q$	Switcheroo

Lesson 40

1.

a. This is valid by the Law of Syllogism.

c. This is valid by Modus Tollens.

e. This is not a valid argument as it exhibits the Fallacy of the Converse.

g. This exhibits the affirmation of the disjunct, and so it is not a valid argument.

2.

a. The argument is invalid, as it exemplifies the Fallacy of the Inverse. Hence, the argument is unsound.

c. By Modus Tollens, the argument is valid. The first premise follows from the definition of a rational number (i.e. a number is rational if it can be expressed as the ratio of two integers). It can also be shown that there are no integers a and b that satisfy $\sqrt{2} = a/b$. Therefore, the argument satisfies the truth condition and is hence sound.

e. This argument is invalid, by the Fallacy of the Inverse. Therefore, it is not sound.

Lesson 41

A.

1. Suppose x and y are odd integers. Then for some integers k_1 and k_2, we can write $x = 2k_1 + 1$ and $y = 2k_2 + 1$. Therefore,

$$x + 2y = (2k_1 + 1) + 2(2k_2 + 1) = 2k_1 + 4k_2 + 3 = 2(k_1 + 2k_2 + 1) + 1.$$

Since $k_1 + 2k_2 + 1$ is an integer, then $x + 2y$ is of the form $2k + 1$, where k is an integer. By Modus Ponens on the definition of odd numbers (the first premise), $x + 2y$ is also an odd integer.

3. Suppose y is even. Then from the first premise, it follows that $y = 2k$ for some integer k. Therefore, $y^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since $2k^2$ is an integer, then y^2 has the form $2m$ for some integer m. Modus Ponens with the first premise leads to the conclusion that y^2 is even.

5. Let l: "Lightning strikes a tree on the farm", h: "All the animals will make sounds of horror", t: "Thunder booms in the sky", and s: "all the animals will run back to the stables".

It can be shown that a counterexample is formed when l, h, and s are true and t is false. Hence, the argument is invalid.

7.

	Proposition	**Reason**
1	$p \wedge q$	Premise
2	p	(1), Simplification
3	$p \vee q$	(2), Addition Law
4	$(p \vee q) \rightarrow r$	Premise
5	r	(3), (4), Modus Ponens

9.

	Proposition	**Reason**
1	$f \rightarrow o$	Premise
2	$(\sim g) \rightarrow (\sim o)$	Premise
3	$o \rightarrow g$	(2), Equivalence to Contraposition
4	$f \rightarrow g$	(1), (3), Law of Syllogism
5	$g \rightarrow v$	Premise
6	$f \rightarrow v$	(4), (5), Law of Syllogism
7	$\sim v$	Premise
8	$\sim f$	(6), Modus Tollens

B.

1. Suppose n is odd and m is even. Then we can write $n = 2k_1 + 1$ and $m = 2k_2$ for some integers k_1 and k_2. Thus,

$$2n + 3m = 2(2k_1 + 1) + 3(2k_2) = 4k_1 + 2 + 6k_2 = 2(2k_1 + 3k_2 + 1).$$

We note that since $2k_1 + 3k_2 + 1$ is an integer, then $2n + 3m$ is even, by the definition of even numbers.

3. (By contradiction) Suppose n is odd. Then we can write $n = 2k + 1$ for some integer k. Therefore,

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Note that since k is an integer, then $2k^2 + 2k$ is also an integer. Therefore, n^2 is odd, which contradicts the premise that n^2 is even. We conclude our assumption is wrong and that n is even.

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