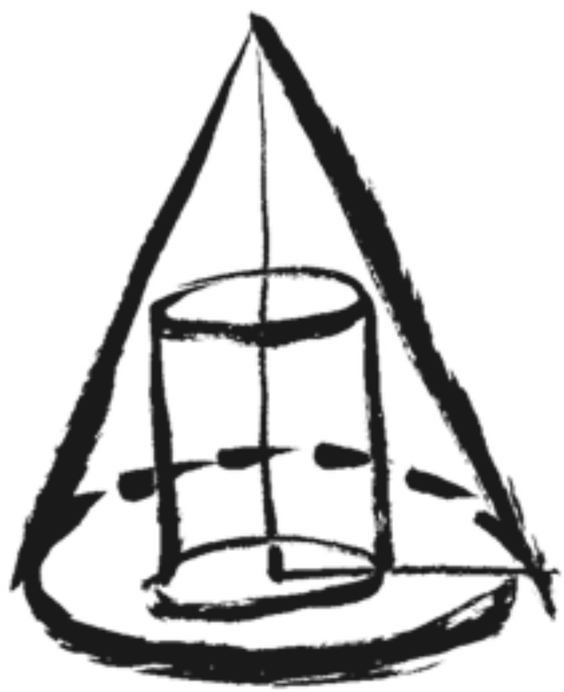
Commission on Higher Education
in collaboration with the Philippine Normal University



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TEACHING GUIDE FOR SENIOR HIGH SCHOOL

Basic Calculus

CORE SUBJECT

This Teaching Guide was collaboratively developed and reviewed by educators from public and private schools, colleges, and universities. We encourage teachers and other education stakeholders to email their feedback, comments, and recommendations to the Commission on Higher Education, K to 12 Transition Program Management Unit - Senior High School Support Team at k12@ched.gov.ph. We value your feedback and recommendations.

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Chairperson: Patricia B. Licuanan, Ph.D.

Commission on Higher Education
K to 12 Transition Program Management Unit
Office Address: 4th Floor, Commission on Higher Education,
C.P. Garcia Ave., Diliman, Quezon City
Telefax: (02) 441-1143 / E-mail Address: k12@ched.gov.ph

DEVELOPMENT TEAM

Team Leader: Jose Maria P. Balmaceda, Ph.D.

Writers:
Carlene Perpetua P. Arceo, Ph.D.
Richard S. Lemence, Ph.D.
Oreste M. Ortega, Jr., M.Sc.
Louie John D. Vallejo, Ph.D.

Technical Editors:
Jose Ernie C. Lope, Ph.D.
Marian P. Roque, Ph.D.

Copy Reader: Roderick B. Lirios

Cover Artists: Paolo Kurtis N. Tan, Renan U. Ortiz

CONSULTANTS

*THIS PROJECT WAS DEVELOPED WITH THE **PHILIPPINE NORMAL UNIVERSITY**.*
University President: Ester B. Ogena, Ph.D.
VP for Academics: Ma. Antoinette C. Montealegre, Ph.D.
VP for University Relations & Advancement: Rosemarievic V. Diaz, Ph.D.

Ma. Cynthia Rose B. Bautista, Ph.D., *CHED*
Bienvenido F. Nebres, S.J., Ph.D., *Ateneo de Manila University*
Carmela C. Oracion, Ph.D., *Ateneo de Manila University*
Minella C. Alarcon, Ph.D., *CHED*
Gareth Price, *Sheffield Hallam University*
Stuart Bevins, Ph.D., *Sheffield Hallam University*

SENIOR HIGH SCHOOL SUPPORT TEAM
CHED K TO 12 TRANSITION PROGRAM MANAGEMENT UNIT

Program Director: Karol Mark R. Yee

Lead for Senior High School Support: Gerson M. Abesamis

Lead for Policy Advocacy and Communications: Averill M. Pizarro

Course Development Officers:
Danie Son D. Gonzalvo, John Carlo P. Fernando

Teacher Training Officers:
Ma. Theresa C. Carlos, Mylene E. Dones

Monitoring and Evaluation Officer: Robert Adrian N. Daulat

Administrative Officers: Ma. Leana Paula B. Bato,
Kevin Ross D. Nera, Allison A. Danao, Ayhen Loisse B. Dalena

Printed in the Philippines by EC-TEC Commercial, No. 32 St. Louis Compound 7, Baesa, Quezon City, ectec_com@yahoo.com

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Introduction

As the Commission supports DepEd's implementation of Senior High School (SHS), it upholds the vision and mission of the K to 12 program, stated in Section 2 of Republic Act 10533, or the Enhanced Basic Education Act of 2013, that "every graduate of basic education be an empowered individual, through a program rooted on...the competence to engage in work and be productive, the ability to coexist in fruitful harmony with local and global communities, the capability to engage in creative and critical thinking, and the capacity and willingness to transform others and oneself."

To accomplish this, the Commission partnered with the Philippine Normal University (PNU), the National Center for Teacher Education, to develop Teaching Guides for Courses of SHS. Together with PNU, this Teaching Guide was studied and reviewed by education and pedagogy experts, and was enhanced with appropriate methodologies and strategies.

Furthermore, the Commission believes that teachers are the most important partners in attaining this goal. Incorporated in this Teaching Guide is a framework that will guide them in creating lessons and assessment tools, support them in facilitating activities and questions, and assist them towards deeper content areas and competencies. Thus, the introduction of the **SHS for SHS Framework.**

The SHS for SHS Framework

The SHS for SHS Framework, which stands for "Saysay-Husay-Sarili for Senior High School," is at the core of this book. The lessons, which combine high-quality content with flexible elements to accommodate diversity of teachers and environments, promote these three fundamental concepts:

SAYSAY: MEANING

Why is this important?

Through this Teaching Guide, teachers will be able to facilitate an understanding of the value of the lessons, for each learner to fully engage in the content on both the cognitive and affective levels.

HUSAY: MASTERY

How will I deeply understand this?

Given that developing mastery goes beyond memorization, teachers should also aim for deep understanding of the subject matter where they lead learners to analyze and synthesize knowledge.

SARILI: OWNERSHIP

What can I do with this?

When teachers empower learners to take ownership of their learning, they develop independence and self-direction, learning about both the subject matter and themselves.

The Parts of the Teaching Guide

This Teaching Guide is mapped and aligned to the DepEd SHS Curriculum, designed to be highly usable for teachers. It contains classroom activities and pedagogical notes, and integrated with innovative pedagogies. All of these elements are presented in the following parts:

1. **INTRODUCTION**
   - Highlight key concepts and identify the essential questions
   - Show the big picture
   - Connect and/or review prerequisite knowledge
   - Clearly communicate learning competencies and objectives
   - Motivate through applications and connections to real-life
2. **INSTRUCTION/DELIVERY**
   - Give a demonstration/lecture/simulation/hands-on activity
   - Show step-by-step solutions to sample problems
   - Use multimedia and other creative tools
   - Give applications of the theory
   - Connect to a real-life problem if applicable
3. **PRACTICE**
   - Discuss worked-out examples
   - Provide easy-medium-hard questions
   - Give time for hands-on unguided classroom work and discovery
   - Use formative assessment to give feedback
4. **ENRICHMENT**
   - Provide additional examples and applications
   - Introduce extensions or generalisations of concepts
   - Engage in reflection questions
   - Encourage analysis through higher order thinking prompts
5. **EVALUATION**
   - Supply a diverse question bank for written work and exercises
   - Provide alternative formats for student work: written homework, journal, portfolio, group/individual projects, student-directed research project

Pedagogical Notes

The teacher should strive to keep a good balance between conceptual understanding and facility in skills and techniques. Teachers are advised to be conscious of the content and performance standards and of the suggested time frame for each lesson, but flexibility in the management of the lessons is possible. Interruptions in the class schedule, or students' poor reception or difficulty with a particular lesson, may require a teacher to extend a particular presentation or discussion.

Computations in some topics may be facilitated by the use of calculators. This is encour- aged; however, it is important that the student understands the concepts and processes involved in the calculation. Exams for the Basic Calculus course may be designed so that calculators are not necessary.

Because senior high school is a transition period for students, the latter must also be prepared for college-level academic rigor. Some topics in calculus require much more rigor and precision than topics encountered in previous mathematics courses, and treatment of the material may be different from teaching more elementary courses. The teacher is urged to be patient and careful in presenting and developing the topics. To avoid too much technical discussion, some ideas can be introduced intuitively and informally, without sacrificing rigor and correctness.

The teacher is encouraged to study the guide very well, work through the examples, and solve exercises, well in advance of the lesson. The development of calculus is one of humankind's greatest achievements. With patience, motivation and discipline, teaching and learning calculus effectively can be realized by anyone. The teaching guide aims to be a valuable resource in this objective.

On DepEd Functional Skills and CHED's College Readiness Standards

As Higher Education Institutions (HEIs) welcome the graduates of the Senior High School program, it is of paramount importance to align Functional Skills set by DepEd with the College Readiness Standards stated by CHED.

The DepEd articulated a set of 21st century skills that should be embedded in the SHS curriculum across various subjects and tracks. These skills are desired outcomes that K to 12 graduates should possess in order to proceed to either higher education, employment, entrepreneurship, or middle-level skills development.

On the other hand, the Commission declared the College Readiness Standards that consist of the combination of knowledge, skills, and reflective thinking necessary to participate and succeed - without remediation - in entry-level undergraduate courses in college.

The alignment of both standards, shown below, is also presented in this Teaching Guide - prepares Senior High School graduates to the revised college curriculum which will initially be implemented by AY 2018-2019.

College Readiness Standards Foundational Skills	DepEd Functional Skills
Produce all forms of texts (written, oral, visual, digital) based on: 1. Solid grounding on Philippine experience and culture; 2. An understanding of the self, community, and nation; 3. Application of critical and creative thinking and doing processes; 4. Competency in formulating ideas/arguments logically, scientifically, and creatively; and 5. Clear appreciation of one's responsibility as a citizen of a multicultural Philippines and a diverse world;	Visual and information literacies Media literacy Critical thinking and problem solving skills Creativity Initiative and self-direction
Systematically apply knowledge, understanding, theory, and skills for the development of the self, local, and global communities using prior learning, inquiry, and experimentation	Global awareness Scientific and economic literacy Curiosity Critical thinking and problem solving skills Risk taking Flexibility and adaptability Initiative and self-direction
Work comfortably with relevant technologies and develop adaptations and innovations for significant use in local and global communities;	Global awareness Media literacy Technological literacy Creativity Flexibility and adaptability Productivity and accountability
Communicate with local and global communities with proficiency, orally, in writing, and through new technologies of communication;	Global awareness Multicultural literacy Collaboration and interpersonal skills Social and cross-cultural skills Leadership and responsibility
Interact meaningfully in a social setting and contribute to the fulfilment of individual and shared goals, respecting the fundamental humanity of all persons and the diversity of groups and communities	Media literacy Multicultural literacy Global awareness Collaboration and interpersonal skills Social and cross-cultural skills Leadership and responsibility Ethical, moral, and spiritual values

Correspondence between the Learning Competencies and the Topics in this Learning Guide

Course Title: Basic Calculus

Semester: Second Semester
No. of Hours/Semester: 80 hrs/sem
Prerequisite: Pre-Calculus

Subject Description: At the end of the course, the students must know how to determine the limit of a function, differentiate, and integrate algebraic, exponential, logarithmic, and trigonometric functions in one variable, and to formulate and solve problems involving continuity, extreme values, related rates, population models, and areas of plane regions.

CONTENT	CONTENT STANDARDS	PERFORMANCE STANDARDS	LEARNING COMPETENCIES	CODE	TOPIC NUMBER
Limits and Continuity	The learners demonstrate an understanding of... the basic concepts of limit and continuity of a function	The learners shall be able to... formulate and solve accurately real-life problems involving continuity of functions	The learners... 1. illustrate the limit of a function using a table of values and the graph of the function	**STEM_BC11LC-IIIa-1**	**1.1**
			2. distinguish between $\lim_{x \to c} f(x)$ and $f(c)$	**STEM_BC11LC-IIIa-2**	**1.2**
			3. illustrate the limit laws	**STEM_BC11LC-IIIa-3**	**1.3**
			4. apply the limit laws in evaluating the limit of algebraic functions (polynomial, rational, and radical)	**STEM_BC11LC-IIIa-4**	**1.4**
			5. compute the limits of exponential, logarithmic,and trigonometric functions using tables of values and graphs of the functions	**STEM_BC11LC-IIIb-1**	**2.1**
			6. evaluate limits involving the expressions $(\sin t)/t$, $(1-\cos t)/t$ and $(e^t - 1)/t$ using tables of values	**STEM_BC11LC-IIIb-2**	**2.2**
			7. illustrate continuity of a function at a number	**STEM_BC11LC-IIIc-1**	**3.1**
			8. determine whether a function is continuous at a number or not	**STEM_BC11LC-IIIc-2**	
			9. illustrate continuity of a function on an interval	**STEM_BC11LC-IIIc-3**	**3.2**
			10. determine whether a function is continuous on an interval or not.	**STEM_BC11LC-IIIc-4**	

CONTENT	CONTENT STANDARDS	PERFORMANCE STANDARDS	LEARNING COMPETENCIES	CODE	TOPIC NUMBER
			11. illustrate different types of discontinuity (hole/removable, jump/essential, asymptotic/infinite)	**STEM_BC11LC-IIId-1**	**4.1**
			12. illustrate the Intermediate Value and Extreme Value Theorems	**STEM_BC11LC-IIId-2**	**4.2**
			13. solves problems involving continuity of a function	**STEM_BC11LC-IIId-3**	**4.3**
Derivatives	basic concepts of derivatives	1. formulate and solve accurately situational problems involving extreme values	1. illustrate the tangent line to the graph of a function at a given point	**STEM_BC11D-IIIe-1**	**5.1**
			2. applies the definition of the derivative of a function at a given number	**STEM_BC11D-IIIe-2**	**5.3**
			3. relate the derivative of a function to the slope of the tangent line	**STEM_BC11D-IIIe-3**	**5.2**
			4. determine the relationship between differentiability and continuity of a function	**STEM_BC11D -IIIf-1**	**6.1**
			5. derive the differentiation rules	**STEM_BC11D-IIIf-2**	**6.2**
			6. apply the differentiation rules in computing the derivative of an algebraic, exponential, and trigonometric functions	**STEM_BC11D-IIIf-3**	
			7. solve optimization problems	**STEM_BC11D-IIIg-1**	**7.1**
		2. formulate and solve accurately situational problems involving related rates	8. compute higher-order derivatives of functions	**STEM_BC11D-IIIh-1**	**8.1**
			9. illustrate the Chain Rule of differentiation	**STEM_BC11D-IIIh-2**	**8.2**
			10. solve problems using the Chain Rule	**STEM_BC11D-IIIh-i-1**	
			11. illustrate implicit differentiation	**STEM_BC11D-IIIi-2**	**9.1**
			12. solve problems (including logarithmic, and inverse trigonometric functions) using implicit differentiation	**STEM_BC11D-IIIi-j-1**	
			13. solve situational problems involving related rates	**STEM_BC11D-IIIj-2**	**10.1**

CONTENT	CONTENT STANDARDS	PERFORMANCE STANDARDS	LEARNING COMPETENCIES	CODE	TOPIC NUMBER
Integration	antiderivatives and Riemann integral	1. formulate and solve accurately situational problems involving population models	1. illustrate an antiderivative of a function	**STEM_BC11I-IVa-1**	**11.1**
			2. compute the general antiderivative of polynomial, radical, exponential, and trigonometric functions	**STEM_BC11I-IVa-b-1**	**11.2-11.4**
			3. compute the antiderivative of a function using substitution rule and table of integrals (includingthose whose antiderivatives involve logarithmic and inverse trigonometric functions)	**STEM_BC11I-IVb-c-1**	**12.1**
			4. solve separable differential equations using antidifferentiation	**STEM_BC11I-IVd-1**	**13.1**
			5. solve situational problems involving exponential growth and decay, bounded growth, and logistic growth	**STEM_BC11I-IVe-f-1**	**14.1**
		2. formulate and solve accurately real-life problems involving areas of plane regions	6. approximate the area of a region under a curve using Riemann sums: (a) left, (b) right, and (c) midpoint	**STEM_BC11I-IVg-1**	**15.1**
			7. define the definite integral as the limit of the Riemann sums	**STEM_BC11I-IVg-2**	**15.2**
			8. illustrate the Fundamental Theorem of Calculus	**STEM_BC11I-IVh-1**	**16.1**
			9. compute the definite integral of a function using the Fundamental Theorem of Calculus	**STEM_BC11I-IVh-2**	**16.2**
			10. illustrates the substitution rule	**STEM_BC11I-IVi-1**	**17.1**
			11. compute the definite integral of a function using the substitution rule	**STEM_BC11I-IVi-2**	
			12. compute the area of a plane region using the definite integral	**STEM_BC11I-IVi-j-1**	**18.1**
			13. solve problems involving areas of plane regions	**STEM_BC11I-IVj-2**	**18.2**

Chapter 1

Limits and Continuity

LESSON 1: The Limit of a Function: Theorems and Examples

TIME FRAME: 4 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate the limit of a function using a table of values and the graph of the function;
2. Distinguish between $\lim\limits_{x \to c} f(x)$ and $f(c)$;
3. Illustrate the limit theorems; and
4. Apply the limit theorems in evaluating the limit of algebraic functions (polynomial, rational, and radical).

LESSON OUTLINE:

1. Evaluation of limits using a table of values
2. Illustrating the limit of a function using the graph of the function
3. Distinguishing between $\lim\limits_{x \to c} f(x)$ and $f(c)$ using a table of values
4. Distinguishing between $\lim\limits_{x \to c} f(x)$ and $f(c)$ using the graph of $y = f(x)$
5. Enumeration of the eight basic limit theorems
6. Application of the eight basic limit theorems on simple examples
7. Limits of polynomial functions
8. Limits of rational functions
9. Limits of radical functions
10. Intuitive notions of infinite limits

TOPIC 1.1: The Limit of a Function

DEVELOPMENT OF THE LESSON

(A) ACTIVITY

In order to find out what the students' idea of a limit is, ask them to bring cutouts of news items, articles, or drawings which for them illustrate the idea of a limit. These may be posted on a wall so that they may see each other's homework, and then have each one explain briefly why they think their particular cutout represents a limit.

(B) INTRODUCTION

Limits are the backbone of calculus, and calculus is called the Mathematics of Change. The study of limits is necessary in studying change in great detail. The evaluation of a particular limit is what underlies the formulation of the derivative and the integral of a function.

For starters, imagine that you are going to watch a basketball game. When you choose seats, you would want to be as close to the action as possible. You would want to be as close to the players as possible and have the best view of the game, as if you were in the basketball court yourself. Take note that you cannot actually be in the court and join the players, but you will be close enough to describe clearly what is happening in the game.

This is how it is with limits of functions. We will consider functions of a single variable and study the behavior of the function as its variable approaches a particular value (a constant). The variable can only take values very, very close to the constant, but it cannot equal the constant itself. However, the limit will be able to describe clearly what is happening to the function near that constant.

(C) LESSON PROPER

Consider a function f of a single variable x. Consider a constant c which the variable x will approach (c may or may not be in the domain of f). The limit, to be denoted by L, is the unique real value that $f(x)$ will approach as x approaches c. In symbols, we write this process as

$$\lim_{x \to c} f(x) = L.$$

This is read, **"The limit of $f(x)$ as x approaches c is L."**

LOOKING AT A TABLE OF VALUES

To illustrate, let us consider

$$\lim_{x\to 2}(1+3x).$$

Here, $f(x) = 1 + 3x$ and the constant c, which x will approach, is 2. To evaluate the given limit, we will make use of a table to help us keep track of the effect that the approach of x toward 2 will have on $f(x)$. Of course, on the number line, x may approach 2 in two ways: through values on its left and through values on its right. We first consider approaching 2 from its left or through values less than 2. Remember that the values to be chosen should be close to 2.

x	$f(x)$
1	4
1.4	5.2
1.7	6.1
1.9	6.7
1.95	6.85
1.997	6.991
1.9999	6.9997
1.9999999	6.9999997

Now we consider approaching 2 from its right or through values greater than but close to 2.

x	$f(x)$
3	10
2.5	8.5
2.2	7.6
2.1	7.3
2.03	7.09
2.009	7.027
2.0005	7.0015
2.0000001	7.0000003

Observe that as the values of x get closer and closer to 2, the values of $f(x)$ get closer and closer to 7. This behavior can be shown no matter what set of values, or what direction, is taken in approaching 2. In symbols,

$$\lim_{x\to 2}(1+3x) = 7.$$

EXAMPLE 1: Investigate

$$\lim_{x \to -1} (x^2 + 1)$$

by constructing tables of values. Here, $c = -1$ and $f(x) = x^2 + 1$.

We start again by approaching -1 from the left.

x	$f(x)$
-1.5	3.25
-1.2	2.44
-1.01	2.0201
-1.0001	2.00020001

Now approach -1 from the right.

x	$f(x)$
-0.5	1.25
-0.8	1.64
-0.99	1.9801
-0.9999	1.99980001

The tables show that as x approaches -1, $f(x)$ approaches 2. In symbols,

$$\lim_{x \to -1} (x^2 + 1) = 2.$$

EXAMPLE 2: Investigate $\lim_{x \to 0} |x|$ through a table of values.

Approaching 0 from the left and from the right, we get the following tables:

x	$\|x\|$
-0.3	0.3
-0.01	0.01
-0.00009	0.00009
-0.00000001	0.00000001

x	$\|x\|$
0.3	0.3
0.01	0.01
0.00009	0.00009
0.00000001	0.00000001

Hence,

$$\lim_{x \to 0} |x| = 0.$$

EXAMPLE 3: Investigate

$$\lim_{x \to 1} \frac{x^2 - 5x + 4}{x - 1}$$

by constructing tables of values. Here, $c = 1$ and $f(x) = \dfrac{x^2 - 5x + 4}{x - 1}$.
Take note that 1 is not in the domain of f, but this is not a problem. In evaluating a limit, remember that we only need to go very close to 1; we will not go to 1 itself.
We now approach 1 from the left.

x	$f(x)$
1.5	-2.5
1.17	-2.83
1.003	-2.997
1.0001	-2.9999

Approach 1 from the right.

x	$f(x)$
0.5	-3.5
0.88	-3.12
0.996	-3.004
0.9999	-3.0001

The tables show that as x approaches 1, $f(x)$ approaches -3. In symbols,

$$\lim_{x \to 1} \frac{x^2 - 5x + 4}{x - 1} = -3.$$

EXAMPLE 4: Investigate through a table of values

$$\lim_{x \to 4} f(x)$$

if

$$f(x) = \begin{cases} x + 1 & \text{if } x < 4 \\ (x - 4)^2 + 3 & \text{if } x \geq 4. \end{cases}$$

This looks a bit different, but the logic and procedure are exactly the same. We still approach the constant 4 from the left and from the right, but note that we should evaluate the appropriate corresponding functional expression. In this case, when x approaches 4 from the left, the values taken should be substituted in $f(x) = x + 1$. Indeed, this is the part of the function which accepts values less than 4. So,

x	$f(x)$
3.7	4.7
3.85	4.85
3.995	4.995
3.99999	4.99999

On the other hand, when x approaches 4 from the right, the values taken should be substituted in $f(x) = (x-4)^2 + 3$. So,

x	$f(x)$
4.3	3.09
4.1	3.01
4.001	3.000001
4.00001	3.0000000001

Observe that the values that $f(x)$ approaches are not equal, namely, $f(x)$ approaches 5 from the left while it approaches 3 from the right. In such a case, we say that the limit of the given function *does not exist* **(DNE)**. In symbols,

$$\lim_{x \to 4} f(x) \text{ DNE.}$$

Remark 1: We need to emphasize an important fact. We do not say that $\lim\limits_{x \to 4} f(x)$ "equals DNE", nor do we write "$\lim\limits_{x \to 4} f(x) = \text{DNE}$", because "DNE" is not a value. In the previous example, "DNE" indicated that the function moves in different directions as its variable approaches c from the left and from the right. In other cases, the limit fails to exist because it is undefined, such as for $\lim\limits_{x \to 0} \frac{1}{x}$ which leads to division of 1 by zero.

Remark 2: Have you noticed a pattern in the way we have been investigating a limit? We have been specifying whether x will approach a value c from the left, through values less than c, or from the right, through values greater than c. This direction may be specified in the limit notation, $\lim\limits_{x \to c} f(x)$ by adding certain symbols.

- If x approaches c from the left, or through values less than c, then we write $\lim\limits_{x \to c^-} f(x)$.
- If x approaches c from the right, or through values greater than c, then we write $\lim\limits_{x \to c^+} f(x)$.

Furthermore, we say

$$\lim_{x \to c} f(x) = L$$

if and only if

$$\lim_{x \to c^-} f(x) = L \text{ and } \lim_{x \to c^+} f(x) = L.$$

In other words, for a limit L to exist, the limits from the left and from the right must both exist and be equal to L. Therefore,

$$\lim_{x\to c} f(x) \text{ DNE whenever } \lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x).$$

These limits, $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$, are also referred to as ***one-sided limits***, since you only consider values on one side of c.

Thus, we may say:

- in our very first illustration that $\lim_{x\to 2}(1+3x) = 7$ because $\lim_{x\to 2^-}(1+3x) = 7$ and $\lim_{x\to 2^+}(1+3x) = 7$.
- in Example 1, $\lim_{x\to -1}(x^2+1) = 2$ since $\lim_{x\to -1^-}(x^2+1) = 2$ and $\lim_{x\to -1^+}(x^2+1) = 2$.
- in Example 2, $\lim_{x\to 0}|x| = 0$ because $\lim_{x\to 0^-}|x| = 0$ and $\lim_{x\to 0^+}|x| = 0$.
- in Example 3, $\lim_{x\to 1}\dfrac{x^2-5x+4}{x-1} = -3$ because $\lim_{x\to 1^-}\dfrac{x^2-5x+4}{x-1} = -3$ and $\lim_{x\to 1^+}\dfrac{x^2-5x+4}{x-1} = -3$.
- in Example 4, $\lim_{x\to 4} f(x)$ DNE because $\lim_{x\to 4^-} f(x) \neq \lim_{x\to 4^+} f(x)$.

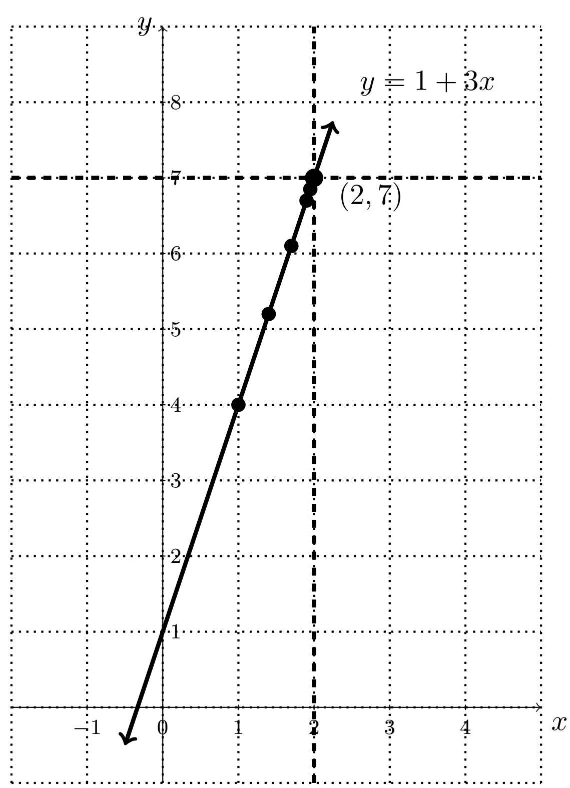
LOOKING AT THE GRAPH OF $y = f(x)$

If one knows the graph of $f(x)$, it will be easier to determine its limits as x approaches given values of c.

Consider again $f(x) = 1+3x$. Its graph is the straight line with slope 3 and intercepts $(0,1)$ and $(-1/3, 0)$. Look at the graph in the vicinity of $x = 2$.

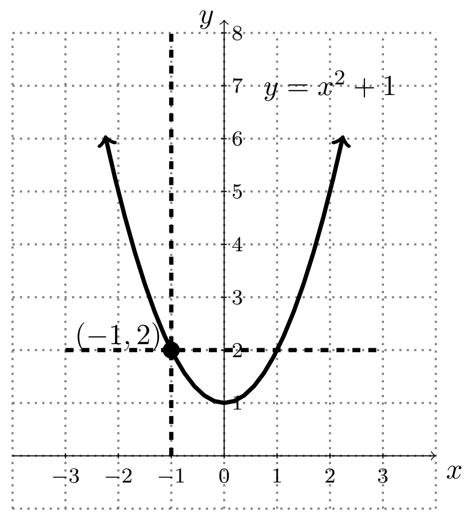
You can easily see the points (from the table of values in page 4) $(1,4)$, $(1.4, 5.2)$, $(1.7, 6.1)$, and so on, approaching the level where $y = 7$. The same can be seen from the right (from the table of values in page 4). Hence, the graph clearly confirms that

$$\lim_{x\to 2}(1+3x) = 7.$$



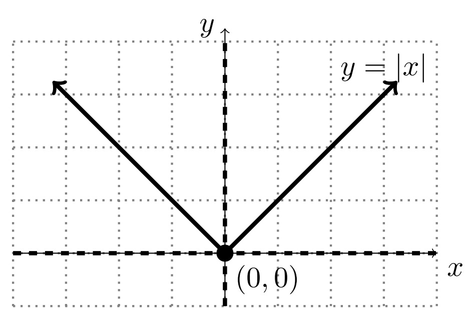
Let us look at the examples again, one by one.

Recall Example 1 where $f(x) = x^2 + 1$. Its graph is given by



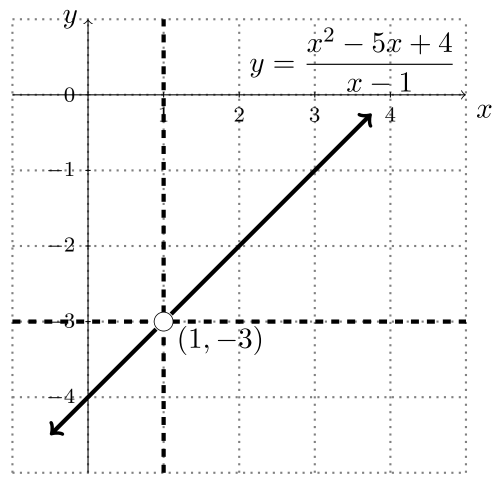
It can be seen from the graph that as values of x approach -1, the values of $f(x)$ approach 2.

Recall Example 2 where $f(x) = |x|$.



It is clear that $\lim_{x \to 0} |x| = 0$, that is, the two sides of the graph both move downward to the origin $(0, 0)$ as x approaches 0.

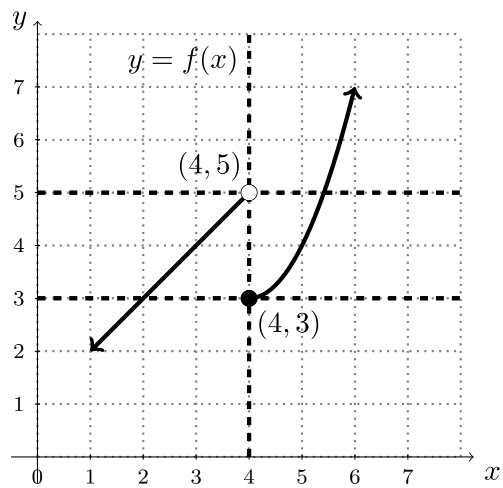
Recall Example 3 where $f(x) = \dfrac{x^2 - 5x + 4}{x - 1}$.



Take note that $f(x) = \dfrac{x^2 - 5x + 4}{x - 1} = \dfrac{(x-4)(x-1)}{x-1} = x - 4$, provided $x \neq 1$. Hence, the graph of $f(x)$ is also the graph of $y = x - 1$, excluding the point where $x = 1$.

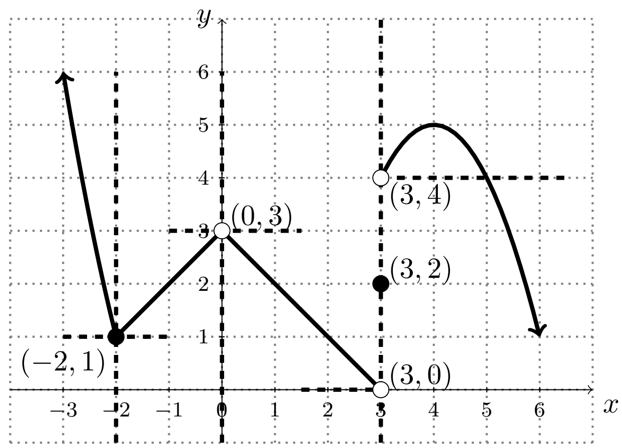
Recall Example 4 where

$$f(x) = \begin{cases} x + 1 & \text{if } x < 4 \\ (x-4)^2 + 3 & \text{if } x \geq 4. \end{cases}$$



Again, we can see from the graph that $f(x)$ has no limit as x approaches 4. The two separate parts of the function move toward different y-levels ($y = 5$ from the left, $y = 3$ from the right) in the vicinity of $c = 4$.

So, in general, if we have the graph of a function, such as below, determining limits can be done much faster and easier by inspection.



For instance, it can be seen from the graph of $y = f(x)$ that:

a. $\lim\limits_{x\to -2} f(x) = 1$.

b. $\lim\limits_{x\to 0} f(x) = 3$. Here, it does not matter that $f(0)$ does not exist (that is, it is undefined, or $x = 0$ is not in the domain of f). Always remember that what matters is the behavior of the function close to $c = 0$ and not precisely at $c = 0$. In fact, even if $f(0)$ were defined and equal to any other constant (not equal to 3), like 100 or -5000, this would still have no bearing on the limit. In cases like this, $\lim\limits_{x\to 0} f(x) = 3$ prevails regardless of the value of $f(0)$, if any.

c. $\lim\limits_{x\to 3} f(x)$ DNE. As can be seen in the figure, the two parts of the graph near $c = 3$ do not move toward a common y-level as x approaches $c = 3$.

(D) **EXERCISES** (Students may use calculators when applicable.)

Exercises marked with a star ($\star$) are challenging problems or may require a longer solution.

1. Complete the following tables of values to investigate $\lim\limits_{x\to 1}(x^2 - 2x + 4)$.

x	$f(x)$
0.5	
0.7	
0.95	
0.995	
0.9995	
0.99995	

x	$f(x)$
1.6	
1.35	
1.05	
1.005	
1.0005	
1.00005	

2. Complete the following tables of values to investigate $\lim\limits_{x\to 0} \dfrac{x-1}{x+1}$.

x	$f(x)$
-1	
-0.8	
-0.35	
-0.1	
-0.09	
-0.0003	
-0.000001	

x	$f(x)$
1	
0.75	
0.45	
0.2	
0.09	
0.0003	
0.000001	

3. Construct a table of values to investigate the following limits:

a. $\lim\limits_{x\to 3} \dfrac{10}{x-2}$

b. $\lim\limits_{x\to 7} \dfrac{10}{x-2}$

c. $\lim\limits_{x\to 2} \dfrac{2x+1}{x-3}$

d. $\lim\limits_{x\to 0} \dfrac{x^2+6}{x^2+2}$

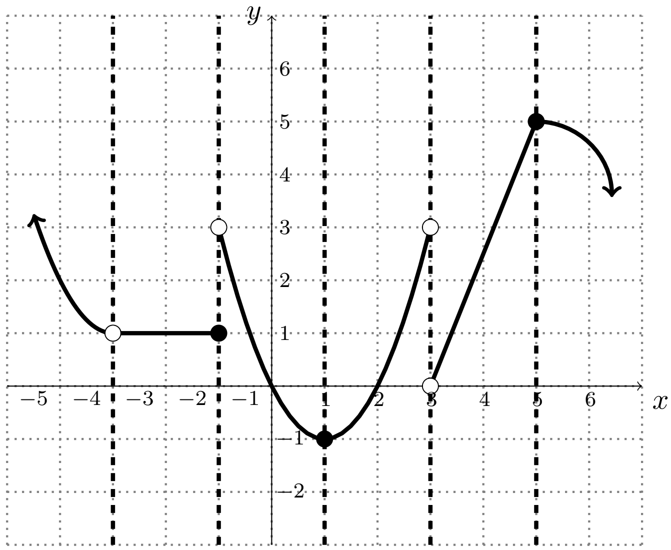
e. $\lim\limits_{x\to 1} \dfrac{1}{x+1}$

f. $\lim\limits_{x\to 0} f(x)$ if $f(x) = \begin{cases} 1/x & \text{if } x \le -1 \\ x^2-2 & \text{if } x > -1 \end{cases}$

g. $\lim\limits_{x\to -1} f(x)$ if $f(x) = \begin{cases} 1/x & \text{if } x \le -1 \\ x^2-2 & \text{if } x > -1 \end{cases}$

h. $\lim\limits_{x\to 1} f(x)$ if $f(x) = \begin{cases} x+3 & \text{if } x < 1 \\ 2x & \text{if } x = 1 \\ \sqrt{5x-1} & \text{if } x > 1 \end{cases}$

4. Consider the function $f(x)$ whose graph is shown below.



Determine the following:

a. $\lim_{x \to -3} f(x)$

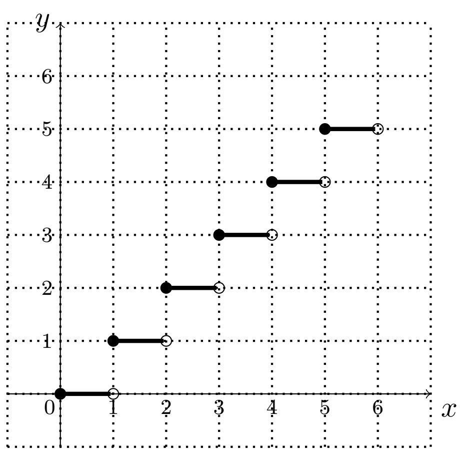
b. $\lim_{x \to -1} f(x)$

c. $\lim_{x \to 1} f(x)$

d. $\lim_{x \to 3} f(x)$

e. $\lim_{x \to 5} f(x)$

5. Consider the function $f(x)$ whose graph is shown below.



What can be said about the limit of $f(x)$

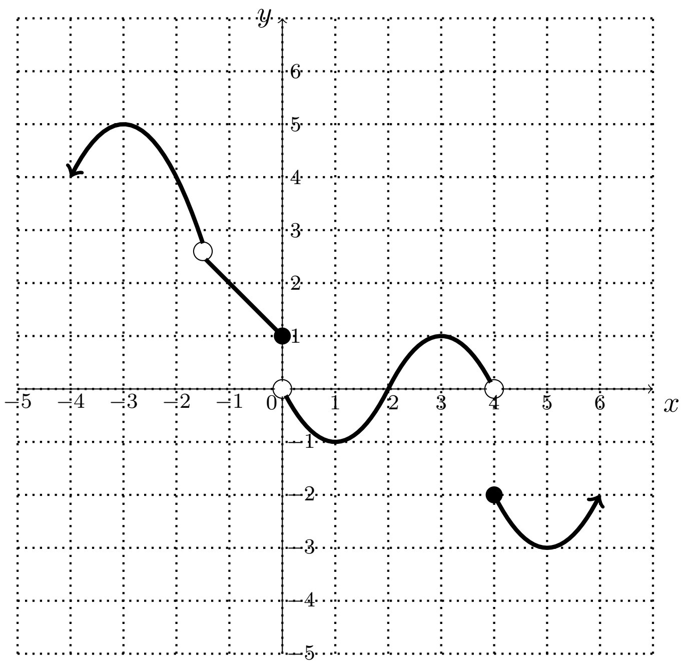
a. at $c = 1,\ 2,\ 3,$ and 4?

b. at integer values of c?

c. at $c = 0.4,\ 2.\ 3, 4.7,$ and 5.5?

d. at non-integer values of c?

6. Consider the function $f(x)$ whose graph is shown below.



Determine the following:

a. $\lim_{x \to -1.5} f(x)$

b. $\lim_{x \to 0} f(x)$

c. $\lim_{x \to 2} f(x)$

d. $\lim_{x \to 4} f(x)$

Teaching Tip

Test how well the students have understood limit evaluation. It is hoped that by now they have observed that *for polynomial and rational functions f*, if c is in the domain of f, then to evaluate $\lim_{x \to c} f(x)$ they just need to substitute the value of c for every x in $f(x)$.

However, this is not true for general functions. Ask the students if they can give an example or point out an earlier example of a case where c is in the domain of f, but $\lim_{x \to c} f(x) \neq f(c)$.

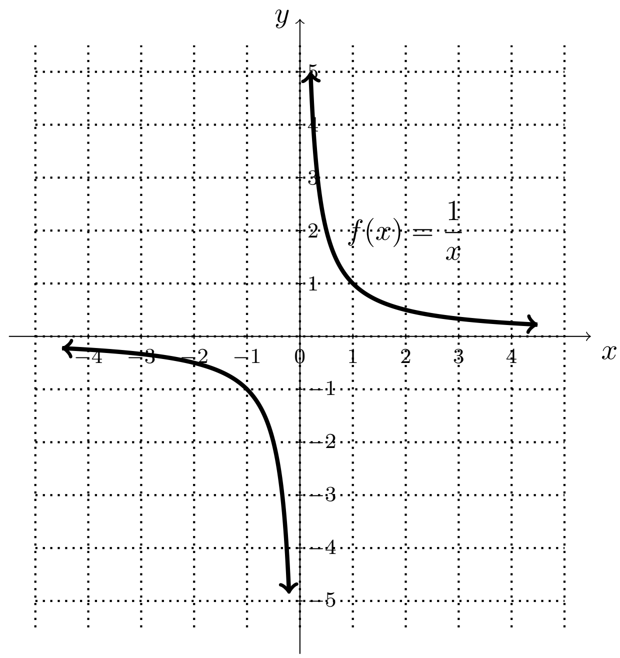
7. Without a table of values and without graphing $f(x)$, give the values of the following limits and explain how you arrived at your evaluation.

a. $\lim_{x \to -1} (3x - 5)$

b. $\lim_{x \to c} \dfrac{x^2 - 9}{x^2 - 4x + 3}$ where $c = 0, 1, 2$

⋆c. $\lim_{x \to 3} \dfrac{x^2 - 9}{x^2 - 4x + 3}$

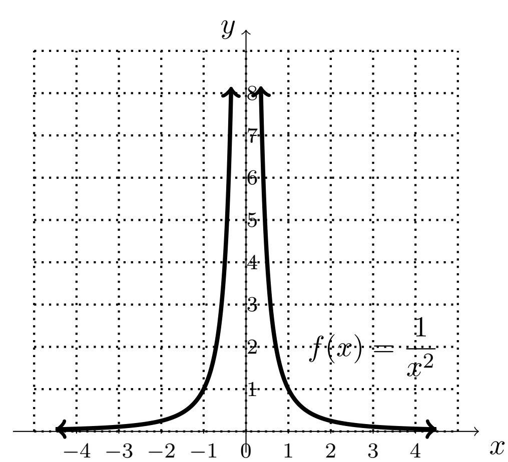
⋆8. Consider the function $f(x) = \dfrac{1}{x}$ whose graph is shown below.



What can be said about $\lim\limits_{x \to 0} f(x)$? Does it exist or not? Why?

Answer: The limit does not exist. From the graph itself, as x-values approach 0, the arrows move in opposite directions. If tables of values are constructed, one for x-values approaching 0 through negative values and another through positive values, it is easy to observe that the closer the x-values are to 0, the more negatively and positively large the corresponding $f(x)$-values become.

⋆9. Consider the function $f(x)$ whose graph is shown below. What can be said about $\lim\limits_{x \to 0} f(x)$? Does it exist or not? Why?



Answer: The limit does not exist. Although as x-values approach 0, the arrows seem to move in the same direction, they will not "stop" at a limiting value. In the absence of such a definite limiting value, we still say the limit does not exist. (We will revisit this function in the lesson about infinite limits where we will discuss more about its behavior near 0.)

⋆10. Sketch one possible graph of a function $f(x)$ defined on $\mathbb{R}$ that satisfies all the listed conditions.

a. $\lim_{x\to 0} f(x) = 1$

b. $\lim_{x\to 1} f(x)$ DNE

c. $\lim_{x\to 2} f(x) = 0$

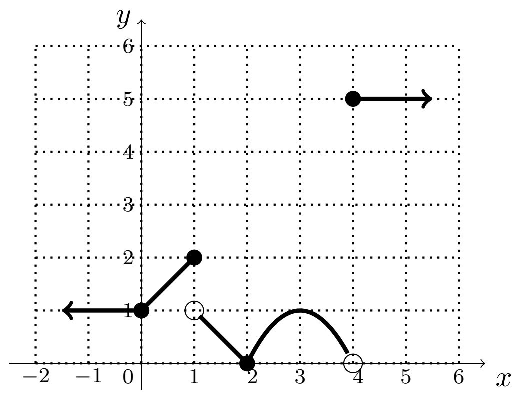
d. $f(1) = 2$

e. $f(2) = 0$

f. $f(4) = 5$

g. $\lim_{x\to c} f(x) = 5$ for all $c > 4$.

Possible answer (there are many other possibilities):



TOPIC 1.2: The Limit of a Function at c versus the Value of the Function at c

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

Critical to the study of limits is the understanding that the value of

$$\lim_{x \to c} f(x)$$

may be distinct from the value of the function at $x = c$, that is, $f(c)$. As seen in previous examples, the limit may be evaluated at values not included in the domain of f. Thus, it must be clear to a student of calculus that the exclusion of a value from the domain of a function does not prohibit the evaluation of the limit of that function at that excluded value, provided of course that f is defined at the points near c. In fact, these cases are actually the more interesting ones to investigate and evaluate.

Furthermore, the awareness of this distinction will help the student understand the concept of continuity, which will be tackled in Lessons 3 and 4.

(B) LESSON PROPER

We will mostly recall our discussions and examples in Lesson 1.
Let us again consider

$$\lim_{x \to 2} (1 + 3x).$$

Recall that its tables of values are:

x	$f(x)$
1	4
1.4	5.2
1.7	6.1
1.9	6.7
1.95	6.85
1.997	6.991
1.9999	6.9997
1.9999999	6.9999997

x	$f(x)$
3	10
2.5	8.5
2.2	7.6
2.1	7.3
2.03	7.09
2.009	7.027
2.0005	7.0015
2.0000001	7.0000003

and we had concluded that $\lim_{x \to 2} (1 + 3x) = 7$.

In comparison, $f(2) = 7$. So, in this example, $\lim_{x \to 2} f(x)$ and $f(2)$ are equal. Notice that the same holds for the next examples discussed:

$\lim_{x \to c} f(x)$	$f(c)$
$\lim_{x \to -1} (x^2 + 1) = 2$	$f(-1) = 2$
$\lim_{x \to 0} \lvert x \rvert = 0$	$f(0) = 0$

This, however, is not always the case. Let us consider the function

$$f(x) = \begin{cases} |x| & \text{if } x \neq 0 \\ 2 & \text{if } x = 0. \end{cases}$$

In contrast to the second example above, the entries are now unequal:

$\lim_{x \to c} f(x)$	$f(c)$
$\lim_{x \to 0} \lvert x \rvert = 0$	$f(0) = 2$

Does this in any way affect the existence of the limit? Not at all. This example shows that $\lim_{x \to c} f(x)$ and $f(c)$ may be distinct.

Furthermore, consider the third example in Lesson 1 where

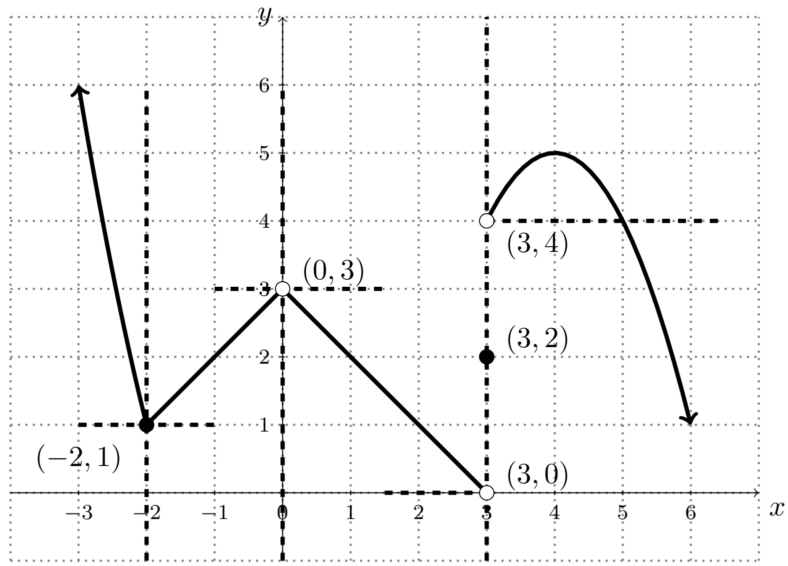
$$f(x) = \begin{cases} x + 1 & \text{if } x < 4 \\ (x-4)^2 + 3 & \text{if } x \geq 4. \end{cases}$$

We have:

$\lim_{x \to c} f(x)$	$f(c)$
$\lim_{x \to 4} f(x)$ DNE	$f(4) = 2$

Once again we see that $\lim_{x \to c} f(x)$ and $f(c)$ are not the same.

A review of the graph given in Lesson 1 (redrawn below) will emphasize this fact.



We restate the conclusions, adding the respective values of $f(c)$:

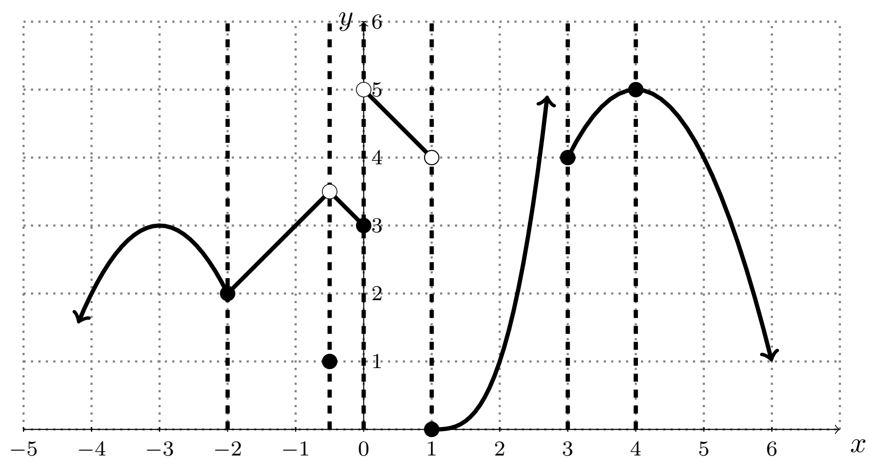
(a) $\lim_{x \to -2} f(x) = 1$ and $f(-2) = 1$.

(b) $\lim_{x \to 0} f(x) = 3$ and $f(0)$ does not exist (or is undefined).

(c) $\lim_{x \to 3} f(x)$ DNE and $f(3)$ also does not exist (or is undefined).

(C) EXERCISES

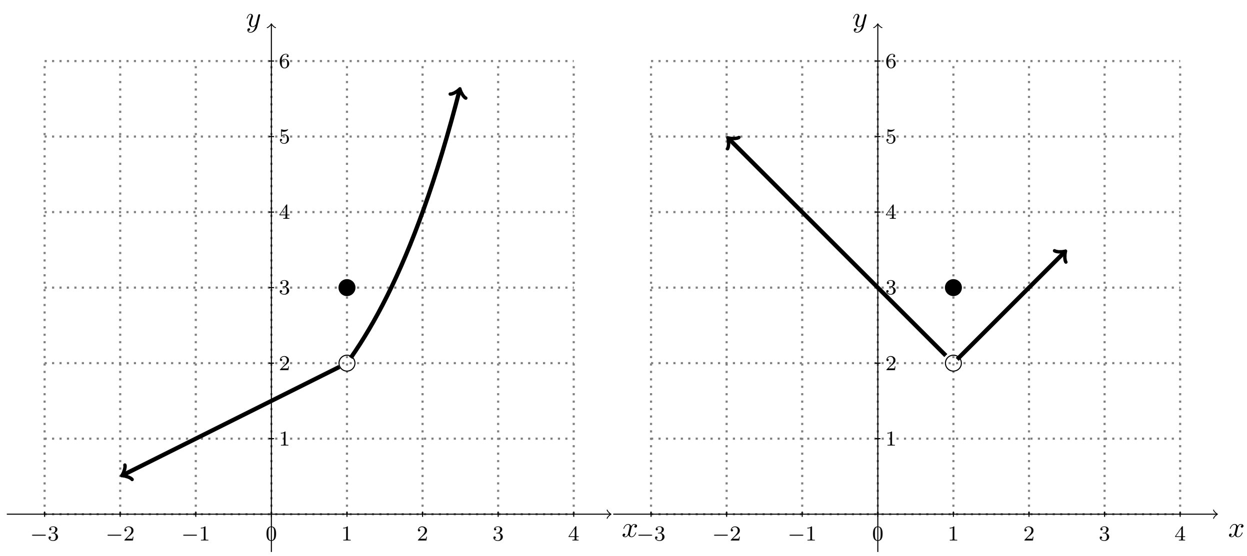
1. Consider the function $f(x)$ whose graph is given below.



Based on the graph, fill in the table with the appropriate values.

c	$\lim_{x\to c} f(x)$	$f(c)$
-2		
$-1/2$		
0		
1		
3		
4		

2. For each given combination of values of $\lim_{x\to c} f(x)$ and $f(c)$, sketch the graph of a possible function that illustrates the combination. For example, if $\lim_{x\to 1} f(x) = 2$ and $f(1) = 3$, then a possible graph of $f(x)$ near $x = 1$ may be any of the two graphs below.



Do a similar rendition for each of the following combinations:

i. $\lim_{x\to 1} f(x) = 2$ and $f(1) = 2$

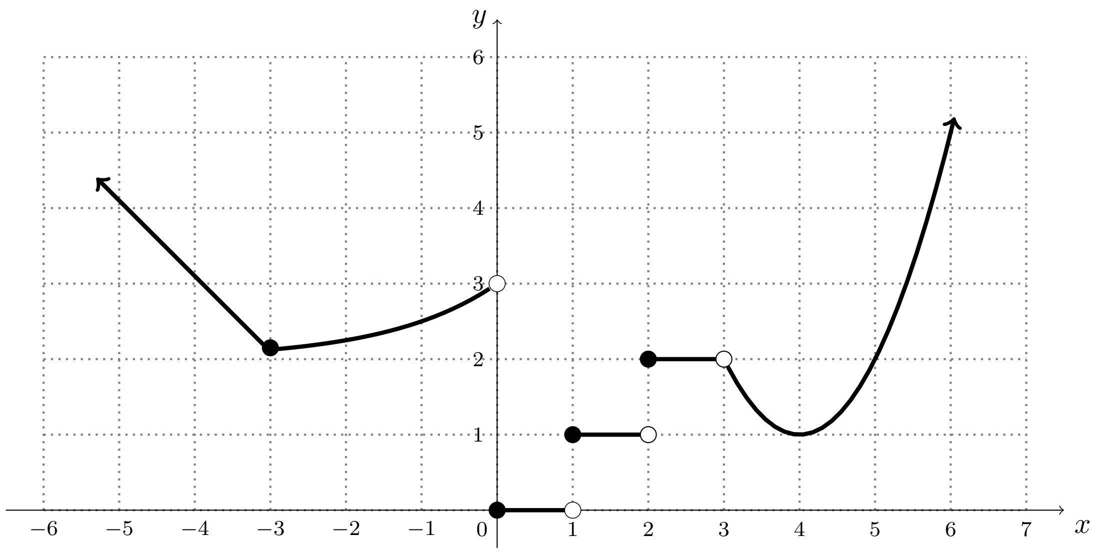
ii. $\lim_{x\to 1} g(x) = -1$ and $g(1) = 1$

iii. $\lim_{x\to 1} h(x)$ DNE and $h(1) = 0$

iv. $\lim_{x\to 1} j(x) = 2$ and $j(1)$ is undefined

v. $\lim_{x\to 1} p(x)$ DNE and $p(1)$ is undefined

3. Consider the function $f(x)$ whose graph is given below.



State whether $\lim_{x \to c} f(x)$ and $f(c)$ are equal or unequal at the given value of c. Also, state whether $\lim_{x \to c} f(x)$ or $f(c)$ does not exist.

i. $c = -3$
ii. $c = -2$
iii. $c = 0$
iv. $c = 0.5$
v. $c = 1$
vi. $c = 2$
vii. $c = 2.3$
viii. $c = 3$
ix. $c = 4$
x. $c = 6$

TOPIC 1.3: Illustration of Limit Theorems

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

Lesson 1 showed us how limits can be determined through either a table of values or the graph of a function. One might ask: Must one always construct a table or graph the function to determine a limit? Filling in a table of values sometimes requires very tedious calculations. Likewise, a graph may be difficult to sketch. However, these should not be reasons for a student to fail to determine a limit.

In this lesson, we will learn how to compute the limit of a function using Limit Theorems.

> **Teaching Tip**
>
> It would be good to recall the parts of Lesson 1 where the students were asked to give the value of a limit, without aid of a table or a graph. Those exercises were intended to lead to the Limit Theorems. These theorems are a formalization of what they had intuitively concluded then.

(B) LESSON PROPER

We are now ready to list down the basic theorems on limits. We will state eight theorems. These will enable us to directly evaluate limits, without need for a table or a graph.

In the following statements, c is a constant, and f and g are functions which may or may not have c in their domains.

1. The limit of a constant is itself. If k is any constant, then,

$$\lim_{x\to c} k = k.$$

For example,

i. $\lim_{x\to c} 2 = 2$
ii. $\lim_{x\to c} -3.14 = -3.14$
iii. $\lim_{x\to c} 789 = 789$

2. The limit of x as x approaches c is equal to c. This may be thought of as the substitution law, because x is simply substituted by c.

$$\lim_{x \to c} x = c.$$

For example,

i. $\lim_{x \to 9} x = 9$

ii. $\lim_{x \to 0.005} x = 0.005$

iii. $\lim_{x \to -10} x = -10$

For the remaining theorems, we will assume that the limits of f and g both exist as x approaches c and that they are L and M, respectively. In other words,

$$\lim_{x \to c} f(x) = L, \quad \text{and} \quad \lim_{x \to c} g(x) = M.$$

3. The Constant Multiple Theorem: This says that the limit of a multiple of a function is simply that multiple of the limit of the function.

$$\lim_{x \to c} k \cdot f(x) = k \cdot \lim_{x \to c} f(x) = k \cdot L.$$

For example, if $\lim_{x \to c} f(x) = 4$, then

i. $\lim_{x \to c} 8 \cdot f(x) = 8 \cdot \lim_{x \to c} f(x) = 8 \cdot 4 = 32.$

ii. $\lim_{x \to c} -11 \cdot f(x) = -11 \cdot \lim_{x \to c} f(x) = -11 \cdot 4 = -44.$

iii. $\lim_{x \to c} \frac{3}{2} \cdot f(x) = \frac{3}{2} \cdot \lim_{x \to c} f(x) = \frac{3}{2} \cdot 4 = 6.$

4. The Addition Theorem: This says that the limit of a sum of functions is the sum of the limits of the individual functions. Subtraction is also included in this law, that is, the limit of a difference of functions is the difference of their limits.

$$\lim_{x \to c}(\ f(x) + g(x)\ ) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = L + M.$$

$$\lim_{x \to c}(\ f(x) - g(x)\ ) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = L - M.$$

For example, if $\lim_{x \to c} f(x) = 4$ and $\lim_{x \to c} g(x) = -5$, then

i. $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = 4 + (-5) = -1.$

ii. $\lim_{x \to c} (f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = 4 - (-5) = 9.$

5. <u>The Multiplication Theorem</u>: This is similar to the Addition Theorem, with multiplication replacing addition as the operation involved. Thus, the limit of a product of functions is equal to the product of their limits.

$$\lim_{x\to c}(f(x)\cdot g(x)) = \lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x) = L\cdot M.$$

Again, let $\lim\limits_{x\to c} f(x) = 4$ and $\lim\limits_{x\to c} g(x) = -5$. Then

$$\lim_{x\to c} f(x)\cdot g(x) = \lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x) = 4\cdot(-5) = -20.$$

Remark 1: The Addition and Multiplication Theorems may be applied to sums, differences, and products of more than two functions.

Remark 2: The Constant Multiple Theorem is a special case of the Multiplication Theorem. Indeed, in the Multiplication Theorem, if the first function $f(x)$ is replaced by a constant k, the result is the Constant Multiple Theorem.

6. <u>The Division Theorem</u>: This says that the limit of a quotient of functions is equal to the quotient of the limits of the individual functions, provided the denominator limit is not equal to 0.

$$\begin{aligned} \lim_{x\to c}\frac{f(x)}{g(x)} &= \frac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)} \\ &= \frac{L}{M}, \quad \text{provided } M \neq 0. \end{aligned}$$

For example,

i. If $\lim\limits_{x\to c} f(x) = 4$ and $\lim\limits_{x\to c} g(x) = -5$,

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)} = \frac{4}{-5} = -\frac{4}{5}.$$

ii. If $\lim\limits_{x\to c} f(x) = 0$ and $\lim\limits_{x\to c} g(x) = -5$,

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{0}{-5} = 0.$$

iii. If $\lim_{x\to c} f(x) = 4$ and $\lim_{x\to c} g(x) = 0$, it is not possible to evaluate $\lim_{x\to c} \frac{f(x)}{g(x)}$, or we may say that the limit DNE.

7. The Power Theorem: This theorem states that the limit of an integer power p of a function is just that power of the limit of the function.

$$\lim_{x\to c}(f(x))^p = (\lim_{x\to c} f(x))^p = L^p.$$

For example,

i. If $\lim_{x\to c} f(x) = 4$, then

$$\lim_{x\to c}(f(x))^3 = (\lim_{x\to c} f(x))^3 = 4^3 = 64.$$

ii. If $\lim_{x\to c} f(x) = 4$, then

$$\lim_{x\to c}(f(x))^{-2} = (\lim_{x\to c} f(x))^{-2} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}.$$

8. The Radical/Root Theorem: This theorem states that if n is a positive integer, the limit of the nth root of a function is just the nth root of the limit of the function, provided the nth root of the limit is a real number. Thus, it is important to keep in mind that if n is even, the limit of the function must be positive.

$$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L}.$$

For example,

i. If $\lim_{x\to c} f(x) = 4$, then

$$\lim_{x\to c} \sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)} = \sqrt{4} = 2.$$

ii. If $\lim_{x\to c} f(x) = -4$, then it is not possible to evaluate $\lim_{x\to c} \sqrt{f(x)}$ because then,

$$\sqrt{\lim_{x\to c} f(x)} = \sqrt{-4},$$

and this is not a real number.

(C) **EXERCISES**

1. Complete the following table.

c	$\lim_{x\to c} 2016$	$\lim_{x\to c} x$
-2		
$-1/2$		
0		
3.1416		
10		
$\sqrt{3}$		

2. Assume the following:

$$\lim_{x\to c} f(x) = \frac{3}{4}, \quad \lim_{x\to c} g(x) = 12, \quad \text{and} \quad \lim_{x\to c} h(x) = -3.$$

Compute the following limits:

a. $\lim_{x\to c}(-4 \cdot f(x))$

b. $\lim_{x\to c} \sqrt{12 \cdot f(x)}$

c. $\lim_{x\to c}(g(x) - h(x))$

d. $\lim_{x\to c}(f(x) \cdot g(x))$

e. $\lim_{x\to c} \dfrac{g(x) + h(x)}{f(x)}$

f. $\lim_{x\to c} \left(\dfrac{f(x)}{h(x)} \cdot g(x)\right)$

g. $\lim_{x\to c}(4 \cdot f(x) + h(x))$

h. $\lim_{x\to c}(8 \cdot f(x) - g(x) - 2 \cdot h(x))$

i. $\lim_{x\to c}(f(x) \cdot g(x) \cdot h(x))$

j. $\lim_{x\to c} \sqrt{-g(x) \cdot h(x)}$

k. $\lim_{x\to c} \dfrac{g(x)}{(h(x))^2}$

l. $\lim_{x\to c} \dfrac{g(x)}{(h(x))^2} \cdot f(x)$

3. Determine whether the statement is True or False. If it is false, explain what makes it false, or provide a counterexample.

a. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then $\lim_{x\to c}(f(x) \pm g(x))$ always exists.

b. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then $\lim_{x\to c}(f(x) \cdot g(x))$ always exists.

c. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ always exists.

d. If $\lim_{x\to c} f(x)$ exists and p is an integer, then $\lim_{x\to c}(f(x))^p$, where p is an integer, always exists.

e. If $\lim_{x\to c} f(x)$ exists and n is a natural number, then $\lim_{x\to c} \sqrt[n]{f(x)}$, always exists.

$\star$f. If $\lim_{x\to c}(f(x) - g(x)) = 0$, then $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ are equal. Answer: False. (Take $f(x) = \dfrac{1}{x} = g(x)$ and $c = 0$.)

$\star$g. If $\lim_{x\to c} \dfrac{f(x)}{g(x)} = 1$, then $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ are equal. Answer: False. (Take $f(x) = \dfrac{1}{x} = g(x)$ and $c = 0$.)

4. Assume the following:

$$\lim_{x\to c} f(x) = 1, \quad \lim_{x\to c} g(x) = -1, \quad \text{and} \quad \lim_{x\to c} h(x) = 2.$$

Compute the following limits:

a. $\lim_{x\to c}(f(x) + g(x))$

b. $\lim_{x\to c}(f(x) - g(x) - h(x))$

c. $\lim_{x\to c}(3 \cdot g(x) + 5 \cdot h(x))$

d. $\lim_{x\to c} \sqrt{f(x)}$

e. $\lim_{x\to c} \sqrt{g(x)}$

f. $\lim_{x\to c} \sqrt[3]{g(x)}$

g. $\lim_{x\to c}(h(x))^5$

h. $\lim_{x\to c} \dfrac{g(x) - f(x)}{h(x)}$

i. $\lim_{x\to c} \left(f(x) \cdot g(x) \cdot (h(x))^2\right)$

j. $\lim_{x\to c} \dfrac{1}{f(x)}$

k. $\lim_{x\to c} \dfrac{1}{g(x)}$

l. $\lim_{x\to c} \dfrac{1}{h(x)}$

m. $\lim_{x\to c} \dfrac{1}{f(x) - h(x)}$

n. $\lim_{x\to c} \dfrac{1}{f(x) + g(x)}$

5. Assume $f(x) = x$. Evaluate

a. $\lim_{x\to 4} f(x)$.

b. $\lim_{x\to 4} \dfrac{1}{f(x)}$.

c. $\lim_{x\to 4} \dfrac{1}{(f(x))^2}$.

d. $\lim_{x\to 4} -\sqrt{f(x)}$.

e. $\lim_{x\to 4} \sqrt{9 \cdot f(x)}$.

f. $\lim_{x\to 4}((f(x))^2 - f(x))$.

g. $\lim_{x\to 4}((f(x))^3 + (f(x))^2 + 2 \cdot f(x))$.

h. $\lim_{x\to 4} \sqrt[n]{3 \cdot (f(x))^2 + 4 \cdot f(x)}$.

i. $\lim_{x\to 4} \dfrac{(f(x))^2 - f(x)}{5 \cdot f(x)}$.

j. $\lim_{x\to 4} \dfrac{(f(x))^2 - 4f(x)}{(f(x))^2 + 4f(x)}$.

TOPIC 1.4: Limits of Polynomial, Rational, and Radical Functions

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

In the previous lesson, we presented and illustrated the limit theorems. We start by recalling these limit theorems.

Theorem 1. *Let c, k, L and M be real numbers, and let $f(x)$ and $g(x)$ be functions defined on some open interval containing c, except possibly at c.*

1. *If $\lim_{x \to c} f(x)$ exists, then it is unique. That is, if $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} f(x) = M$, then $L = M$.*
2. $\lim_{x \to c} c = c$.
3. $\lim_{x \to c} x = c$
4. *Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$.*
   - i. *(Constant Multiple)* $\lim_{x \to c} [k \cdot g(x)] = k \cdot M$.
   - ii. *(Addition)* $\lim_{x \to c} [f(x) \pm g(x)] = L \pm M$.
   - iii. *(Multiplication)* $\lim_{x \to c} [f(x)g(x)] = LM$.
   - iv. *(Division)* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, *provided $M \neq 0$.*
   - v. *(Power)* $\lim_{x \to c} [f(x)]^p = L^p$ *for p, a positive integer.*
   - vi. *(Root/Radical)* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$ *for positive integers n, and provided that $L > 0$ when n is even.*

Teaching Tip

It would be helpful for the students if these limit theorems remain written on the board or on manila paper throughout the discussion of this lesson.

In this lesson, we will show how these limit theorems are used in evaluating algebraic functions. Particularly, we will illustrate how to use them to evaluate the limits of polynomial, rational and radical functions.

(B) LESSON PROPER

LIMITS OF ALGEBRAIC FUNCTIONS

We start with evaluating the limits of polynomial functions.

EXAMPLE 1: Determine $\lim_{x \to 1}(2x+1)$.

Solution. From the theorems above,

$$\begin{aligned}
\lim_{x \to 1}(2x+1) &= \lim_{x \to 1} 2x + \lim_{x \to 1} 1 && \text{(Addition)} \\
&= \left(2 \lim_{x \to 1} x\right) + 1 && \text{(Constant Multiple)} \\
&= 2(1) + 1 && \left(\lim_{x \to c} x = c\right) \\
&= 2 + 1 \\
&= 3.
\end{aligned}$$

EXAMPLE 2: Determine $\lim_{x \to -1}(2x^3 - 4x^2 + 1)$.

Solution. From the theorems above,

$$\begin{aligned}
\lim_{x \to -1}(2x^3 - 4x^2 + 1) &= \lim_{x \to -1} 2x^3 - \lim_{x \to -1} 4x^2 + \lim_{x \to -1} 1 && \text{(Addition)} \\
&= 2 \lim_{x \to -1} x^3 - 4 \lim_{x \to -1} x^2 + 1 && \text{(Constant Multiple)} \\
&= 2(-1)^3 - 4(-1)^2 + 1 && \text{(Power)} \\
&= -2 - 4 + 1 \\
&= -5.
\end{aligned}$$

EXAMPLE 3: Evaluate $\lim_{x \to 0}(3x^4 - 2x - 1)$.

Solution. From the theorems above,

$$\begin{aligned}
\lim_{x \to 0}(3x^4 - 2x - 1) &= \lim_{x \to 0} 3x^4 - \lim_{x \to 0} 2x - \lim_{x \to 0} 1 && \text{(Addition)} \\
&= 3 \lim_{x \to 0} x^4 - 2 \lim_{x \to 0} x^2 - 1 && \text{(Constant Multiple)} \\
&= 3(0)^4 - 2(0) - 1 && \text{(Power)} \\
&= 0 - 0 - 1 \\
&= -1.
\end{aligned}$$

We will now apply the limit theorems in evaluating rational functions. In evaluating the limits of such functions, recall from Theorem 1 the Division Rule, and all the rules stated in Theorem 1 which have been useful in evaluating limits of polynomial functions, such as the Addition and Product Rules.

EXAMPLE 4: Evaluate $\lim_{x\to 1} \frac{1}{x}$.

Solution. First, note that $\lim_{x\to 1} x = 1$. Since the limit of the denominator is nonzero, we can apply the Division Rule. Thus,

$$\begin{aligned} \lim_{x\to 1} \frac{1}{x} &= \frac{\lim_{x\to 1} 1}{\lim_{x\to 1} x} && \text{(Division)} \\ &= \frac{1}{1} \\ &= 1. \end{aligned}$$

EXAMPLE 5: Evaluate $\lim_{x\to 2} \frac{x}{x+1}$.

Solution. We start by checking the limit of the polynomial function in the denominator.

$$\lim_{x\to 2}(x+1) = \lim_{x\to 2} x + \lim_{x\to 2} 1 = 2 + 1 = 3.$$

Since the limit of the denominator is not zero, it follows that

$$\lim_{x\to 2} \frac{x}{x+1} = \frac{\lim_{x\to 2} x}{\lim_{x\to 2}(x+1)} = \frac{2}{3} \qquad \text{(Division)}$$

EXAMPLE 6: Evaluate $\lim_{x\to 1} \frac{(x-3)(x^2-2)}{x^2+1}$. First, note that

$$\lim_{x\to 1}(x^2+1) = \lim_{x\to 1} x^2 + \lim_{x\to 1} 1 = 1 + 1 = 2 \neq 0.$$

Thus, using the theorem,

$$\begin{aligned} \lim_{x\to 1} \frac{(x-3)(x^2-2)}{x^2+1} &= \frac{\lim_{x\to 1}(x-3)(x^2-2)}{\lim_{x\to 1}(x^2+1)} && \text{(Division)} \\ &= \frac{\lim_{x\to 1}(x-3) \cdot \lim_{x\to 1}(x^2-2)}{2} && \text{(Multipication)} \\ &= \frac{\left(\lim_{x\to 1} x - \lim_{x\to 1} 3\right)\left(\lim_{x\to 1} x^2 - \lim_{x\to 1} 2\right)}{2} && \text{(Addition)} \\ &= \frac{(1-3)(1^2-2)}{2} \\ &= 1. \end{aligned}$$

Theorem 2. *Let f be a polynomial of the form*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0.$$

If c is a real number, then

$$\lim_{x\to c} f(x) = f(c).$$

Proof. Let c be any real number. Remember that a polynomial is defined at any real number. So,

$$f(c) = a_n c^n + a_{n-1}c^{n-1} + a_{n-2}c^{n-2} + ... + a_1 c + a_0.$$

Now apply the limit theorems in evaluating $\lim\limits_{x\to c} f(x)$:

$$\begin{aligned}
\lim_{x\to c} f(x) &= \lim_{x\to c}(a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0) \\
&= \lim_{x\to c} a_n x^n + \lim_{x\to c} a_{n-1}x^{n-1} + \lim_{x\to c} a_{n-2}x^{n-2} + ... + \lim_{x\to c} a_1 x + \lim_{x\to c} a_0 \\
&= a_n \lim_{x\to c} x^n + a_{n-1} \lim_{x\to c} x^{n-1} + a_{n-2} \lim_{x\to c} x^{n-2} + ... + a_1 \lim_{x\to c} x + a_0 \\
&= a_n c^n + a_{n-1}c^{n-1} + a_{n-2}c^{n-2} + ... + a_1 c + a_0 \\
&= f(c).
\end{aligned}$$

Therefore, $\lim\limits_{x\to c} f(x) = f(c)$. □

EXAMPLE 7: Evaluate $\lim\limits_{x\to -1}(2x^3 - 4x^2 + 1)$.

Solution. Note first that our function

$$f(x) = 2x^3 - 4x^2 + 1,$$

is a polynomial. Computing for the value of f at $x = -1$, we get

$$f(-1) = 2(-1)^3 - 4(-1)^2 + 1 = 2(-1) - 4(1) + 1 = -5.$$

Therefore, from Theorem 2,

$$\lim_{x\to -1}(2x^3 - 4x^2 + 1) = f(-1) = -5.$$

Note that we get the same answer when we use limit theorems.

Theorem 3. *Let h be a rational function of the form $h(x) = \dfrac{f(x)}{g(x)}$ where f and g are polynomial functions. If c is a real number and $g(c) \neq 0$, then*

$$\lim_{x\to c} h(x) = \lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}.$$

Proof. From Theorem 2, $\lim_{x \to c} g(x) = g(c)$, which is nonzero by assumption. Moreover, $\lim_{x \to c} f(x) = f(c)$. Therefore, by the Division Rule of Theorem 1,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{f(c)}{g(c)}.$$

$\square$

EXAMPLE 8: Evaluate $\lim_{x \to 1} \frac{1 - 5x}{1 + 3x^2 + 4x^4}$.

Solution. Since the denominator is not zero when evaluated at $x = 1$, we may apply Theorem 3:

$$\lim_{x \to 1} \frac{1 - 5x}{1 + 3x^2 + 4x^4} = \frac{1 - 5(1)}{1 + 3(1)^2 + 4(1)^4} = \frac{-4}{8} = -\frac{1}{2}.$$

We will now evaluate limits of radical functions using limit theorems.

EXAMPLE 9: Evaluate $\lim_{x \to 1} \sqrt{x}$.

Solution. Note that $\lim_{x \to 1} x = 1 > 0$. Therefore, by the Radical/Root Rule,

$$\lim_{x \to 1} \sqrt{x} = \sqrt{\lim_{x \to 1} x} = \sqrt{1} = 1.$$

EXAMPLE 10: Evaluate $\lim_{x \to 0} \sqrt{x + 4}$.

Solution. Note that $\lim_{x \to 0} (x + 4) = 4 > 0$. Hence, by the Radical/Root Rule,

$$\lim_{x \to 0} \sqrt{x + 4} = \sqrt{\lim_{x \to 0} (x + 4)} = \sqrt{4} = 2.$$

EXAMPLE 11: Evaluate $\lim_{x \to -2} \sqrt[3]{x^2 + 3x - 6}$.

Solution. Since the index of the radical sign is odd, we do not have to worry that the limit of the radicand is negative. Therefore, the Radical/Root Rule implies that

$$\lim_{x \to -2} \sqrt[3]{x^2 + 3x - 6} = \sqrt[3]{\lim_{x \to -2} (x^2 + 3x - 6)} = \sqrt[3]{4 - 6 - 6} = \sqrt[3]{-8} = -2.$$

EXAMPLE 12: Evaluate $\lim\limits_{x\to 2} \dfrac{\sqrt{2x+5}}{1-3x}$.

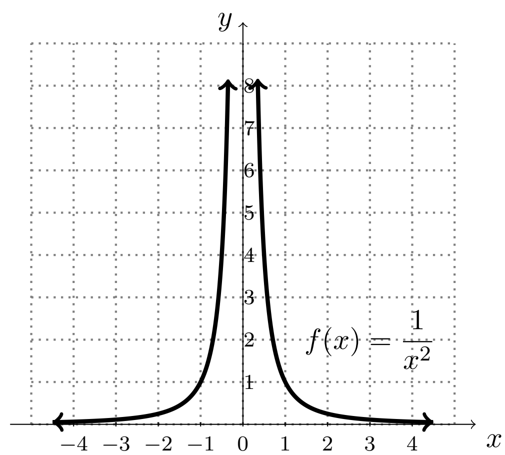
***Solution*.** First, note that $\lim\limits_{x\to 2}(1-3x) = -5 \neq 0$. Moreover, $\lim\limits_{x\to 2}(2x+5) = 9 > 0$. Thus, using the Division and Radical Rules of Theorem 1, we obtain

$$\lim_{x\to 2} \frac{\sqrt{2x+5}}{1-3x} = \frac{\lim\limits_{x\to 2}\sqrt{2x+5}}{\lim\limits_{x\to 2} 1-3x} = \frac{\sqrt{\lim\limits_{x\to 2}(2x+5)}}{-5} = \frac{\sqrt{9}}{-5} = -\frac{3}{5}.$$

INTUITIVE NOTIONS OF INFINITE LIMITS

We investigate the limit at a point c of a rational function of the form $\dfrac{f(x)}{g(x)}$ where f and g are polynomial functions with $f(c) \neq 0$ and $g(c) = 0$. Note that Theorem 3 does not cover this because it assumes that the denominator is nonzero at c.

Now, consider the function $f(x) = \dfrac{1}{x^2}$. Note that the function is not defined at $x = 0$ but we can check the behavior of the function as x approaches 0 intuitively. We first consider approaching 0 from the left.



x	$f(x)$
-0.9	1.2345679
-0.5	4
-0.1	100
-0.01	$10,000$
-0.001	$1,000,000$
-0.0001	$100,000,000$

Observe that as x approaches 0 from the left, the value of the function increases without bound. When this happens, we say that the limit of $f(x)$ as x approaches 0 from the left is *positive infinity*, that is,

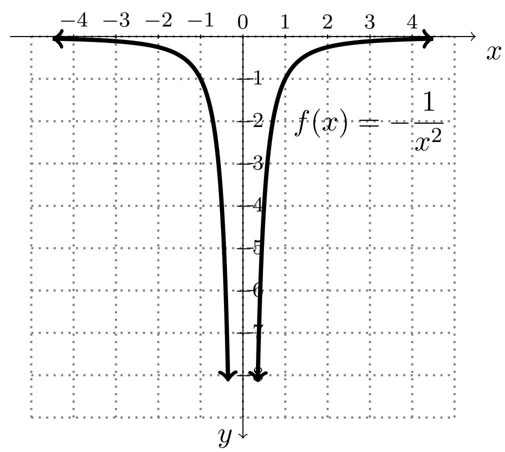
$$\lim_{x\to 0^-} f(x) = +\infty.$$

x	$f(x)$
0.9	1.2345679
0.5	4
0.1	100
0.01	$10,000$
0.001	$1,000,000$
0.0001	$100,000,000$

Again, as x approaches 0 from the right, the value of the function increases without bound, so, $\lim_{x\to 0^+} f(x) = +\infty$.

Since $\lim_{x\to 0^-} f(x) = +\infty$ and $\lim_{x\to 0^+} f(x) = +\infty$, we may conclude that $\lim_{x\to 0} f(x) = +\infty$.

Now, consider the function $f(x) = -\dfrac{1}{x^2}$. Note that the function is not defined at $x = 0$ but we can still check the behavior of the function as x approaches 0 intuitively. We first consider approaching 0 from the left.



x	$f(x)$
-0.9	-1.2345679
-0.5	-4
-0.1	-100
-0.01	$-10,000$
-0.001	$-1,000,000$
-0.0001	$-100,000,000$

This time, as x approaches 0 from the left, the value of the function decreases without bound. So, we say that the limit of $f(x)$ as x approaches 0 from the left is negative infinity, that is,

$$\lim_{x\to 0^-} f(x) = -\infty.$$

x	$f(x)$
0.9	-1.2345679
0.5	-4
0.1	-100
0.01	$-10,000$
0.001	$-1,000,000$
0.0001	$-100,000,000$

As x approaches 0 from the right, the value of the function also decreases without bound, that is, $\lim_{x\to 0^+} f(x) = -\infty$.

Since $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = -\infty$, we are able to conclude that $\lim_{x\to 0} f(x) = -\infty$.

We now state the intuitive definition of *infinite limits* of functions:

The limit of $f(x)$ as x approaches c is positive infinity, denoted by,

$$\lim_{x\to c} f(x) = +\infty$$

if the value of $f(x)$ increases without bound whenever the values of x get closer and closer to c. The limit of $f(x)$ as x approaches c is negative infinity, denoted by,

$$\lim_{x\to c} f(x) = -\infty$$

if the value of $f(x)$ decreases without bound whenever the values of x get closer and closer to c.

Let us consider $f(x) = \dfrac{1}{x}$. The graph on the right suggests that

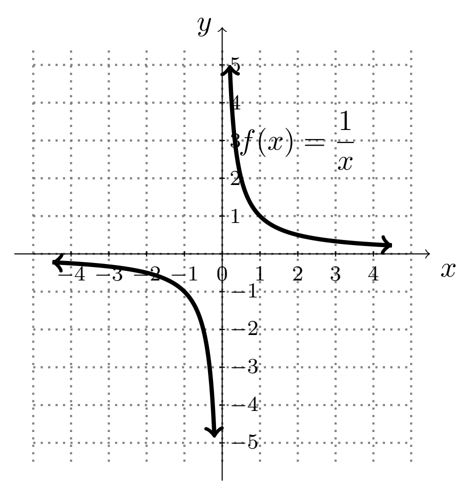
$$\lim_{x\to 0^-} f(x) = -\infty$$

while

$$\lim_{x\to 0^+} f(x) = +\infty.$$

Because the one-sided limits are not the same, we say that

$$\lim_{x\to 0} f(x) \text{ DNE}.$$



Remark 1: Remember that ∞ is NOT a number. It holds no specific value. So, $\lim_{x\to c} f(x) = +\infty$ or $\lim_{x\to c} f(x) = -\infty$ describes the behavior of the function near $x = c$, but it does not exist as a real number.

Remark 2: Whenever $\lim\limits_{x\to c^+} f(x) = \pm\infty$ or $\lim\limits_{x\to c^-} f(x) = \pm\infty$, we normally see the dashed vertical line $x = c$. This is to indicate that the graph of $y = f(x)$ is *asymptotic* to $x = c$, meaning, the graphs of $y = f(x)$ and $x = c$ are very close to each other near c. In this case, we call $x = c$ a ***vertical asymptote*** of the graph of $y = f(x)$.

Teaching Tip

Computing infinite limits is not a learning objective of this course, however, we will be needing this notion for the discussion on infinite essential discontinuity, which will be presented in Topic 4.1. It is enough that the student determines that the limit at the point c is $+\infty$ or $-\infty$ from the behavior of the graph, or the trend of the y-coordinates in a table of values.

(C) **EXERCISES**

I. Evaluate the following limits.

1. $\lim\limits_{w\to 1}(1+\sqrt[3]{w})(2-w^2+3w^3)$
2. $\lim\limits_{t\to -2}\dfrac{t^2-1}{t^2+3t-1}$
3. $\lim\limits_{z\to 2}\left(\dfrac{2z+z^2}{z^2+4}\right)^3$
4. $\lim\limits_{x\to 0}\dfrac{x^2-x-2}{x^3-6x^2-7x+1}$
5. $\lim\limits_{y\to -2}\dfrac{4-3y^2-y^3}{6-y-y^2}$
6. $\lim\limits_{x\to -1}\dfrac{x^3-7x^2+14x-8}{2x^2-3x-4}$
7. $\lim\limits_{x\to -1}\dfrac{\sqrt{x^2+3}-2}{x^2+1}$
8. $\lim\limits_{x\to 2}\dfrac{\sqrt{2x}-\sqrt{6-x}}{4+x^2}$

II. Complete the following tables.

x	$\dfrac{x-5}{x-3}$	$\dfrac{x}{x^2-6x+9}$
2.5		
2.8		
2.9		
2.99		
2.999		
2.9999		

x	$\dfrac{x-5}{x-3}$	$\dfrac{x}{x^2-6x+9}$
3.5		
3.2		
3.1		
3.01		
3.001		
3.0001		

From the table, determine the following limits.

1. $\lim\limits_{x\to 3^-}\dfrac{x-5}{x-3}$
2. $\lim\limits_{x\to 3^+}\dfrac{x-5}{x-3}$
3. $\lim\limits_{x\to 3}\dfrac{x-5}{x-3}$
4. $\lim\limits_{x\to 3^-}\dfrac{x}{x^2-6x+9}$
5. $\lim\limits_{x\to 3^+}\dfrac{x}{x^2-6x+9}$
6. $\lim\limits_{x\to 3}\dfrac{x}{x^2-6x+9}$

III. Recall the graph of $y = \csc x$. From the behavior of the graph of the cosecant function, determine if the following limits evaluate to $+\infty$ or to $-\infty$.

1. $\lim\limits_{x \to 0^-} \csc x$
2. $\lim\limits_{x \to 0^+} \csc x$
3. $\lim\limits_{x \to \pi^-} \csc x$
4. $\lim\limits_{x \to \pi^+} \csc x$

IV. Recall the graph of $y = \tan x$.

1. Find the value of $c \in (0, \pi)$ such that $\lim\limits_{x \to c^-} \tan x = +\infty$.
2. Find the value of $d \in (\pi, 2\pi)$ such that $\lim\limits_{x \to d^+} \tan x = -\infty$.

LESSON 2: Limits of Some Transcendental Functions and Some Indeterminate Forms

TIME FRAME: 4 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Compute the limits of exponential, logarithmic, and trigonometric functions using tables of values and graphs of the functions;
2. Evaluate the limits of expressions involving $\frac{\sin t}{t}$, $\frac{1-\cos t}{t}$, and $\frac{e^t-1}{t}$ using tables of values; and
3. Evaluate the limits of expressions resulting in the indeterminate form $\frac{\text{“}0\text{”}}{0}$.

LESSON OUTLINE:

1. Exponential functions
2. Logarithmic functions
3. Trigonometric functions
4. Evaluating $\lim\limits_{t\to 0} \frac{\sin t}{t}$
5. Evaluating $\lim\limits_{t\to 0} \frac{1-\cos t}{t}$
6. Evaluating $\lim\limits_{t\to 0} \frac{e^t-1}{t}$
7. Indeterminate form $\frac{\text{“}0\text{”}}{0}$

TOPIC 2.1: Limits of Exponential, Logarithmic, and Trigonometric Functions

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

Real-world situations can be expressed in terms of functional relationships. These functional relationships are called mathematical models. In applications of calculus, it is quite important that one can generate these mathematical models. They sometimes use functions that you encountered in precalculus, like the exponential, logarithmic, and trigonometric functions. Hence, we start this lesson by recalling these functions and their corresponding graphs.

(a) If $b > 0$, $b \neq 1$, the *exponential function with base b* is defined by

$$f(x) = b^x, \ x \in \mathbb{R}.$$

(b) Let $b > 0, b \neq 1$. If $b^y = x$ then y is called the *logarithm of x to the base b*, denoted $y = \log_b x$.

> **Teaching Tip**
>
> Allow students to use their calculators.

(B) LESSON PROPER

EVALUATING LIMITS OF EXPONENTIAL FUNCTIONS

First, we consider the natural exponential function $f(x) = e^x$, where e is called the *Euler number*, and has value 2.718281....

EXAMPLE 1: Evaluate the $\lim_{x \to 0} e^x$.

Solution. We will construct the table of values for $f(x) = e^x$. We start by approaching the number 0 from the left or through the values less than but close to 0.

> **Teaching Tip**
>
> Some students may not be familiar with the natural number e on their scientific calculators. Demonstrate to them how to properly input powers of e on their calculators .

x	$f(x)$
-1	0.36787944117
-0.5	0.60653065971
-0.1	0.90483741803
-0.01	0.99004983374
-0.001	0.99900049983
-0.0001	0.999900049983
-0.00001	0.99999000005

Intuitively, from the table above, $\lim_{x \to 0^-} e^x = 1$. Now we consider approaching 0 from its right or through values greater than but close to 0.

x	$f(x)$
1	2.71828182846
0.5	1.6487212707
0.1	1.10517091808
0.01	1.01005016708
0.001	1.00100050017
0.0001	1.000100005
0.00001	1.00001000005

From the table, as the values of x get closer and closer to 0, the values of $f(x)$ get closer and closer to 1. So, $\lim_{x \to 0^+} e^x = 1$. Combining the two one-sided limits allows us to conclude that

$$\lim_{x \to 0} e^x = 1.$$

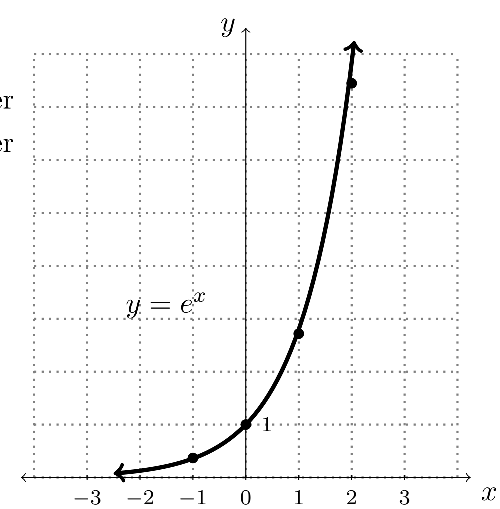
We can use the graph of $f(x) = e^x$ to determine its limit as x approaches 0. The figure below is the graph of $f(x) = e^x$.

Looking at Figure 1.1, as the values of x approach 0, either from the right or the left, the values of $f(x)$ will get closer and closer to 1. We also have the following:

(a) $\lim_{x \to 1} e^x = e = 2.718...$

(b) $\lim_{x \to 2} e^x = e^2 = 7.389...$

(c) $\lim_{x \to -1} e^x = e^{-1} = 0.367...$



EVALUATING LIMITS OF LOGARITHMIC FUNCTIONS

Now, consider the natural logarithmic function $f(x) = \ln x$. Recall that $\ln x = \log_e x$. Moreover, it is the inverse of the natural exponential function $y = e^x$.

EXAMPLE 2: Evaluate $\lim\limits_{x \to 1} \ln x$.

Solution. We will construct the table of values for $f(x) = \ln x$. We first approach the number 1 from the left or through values less than but close to 1.

x	$f(x)$
0.1	-2.30258509299
0.5	-0.69314718056
0.9	-0.10536051565
0.99	-0.01005033585
0.999	-0.00100050033
0.9999	-0.000100005
0.99999	-0.00001000005

Intuitively, $\lim\limits_{x \to 1^-} \ln x = 0$. Now we consider approaching 1 from its right or through values greater than but close to 1.

x	$f(x)$
2	0.69314718056
1.5	0.4054651081
1.1	0.0953101798
1.01	0.00995033085
1.001	0.00099950033
1.0001	0.000099995
1.00001	0.00000999995

Intuitively, $\lim\limits_{x \to 1^+} \ln x = 0$. As the values of x get closer and closer to 1, the values of $f(x)$ get closer and closer to 0. In symbols,

$$\lim_{x \to 1} \ln x = 0.$$

We now consider the *common logarithmic function* $f(x) = \log_{10} x$. Recall that $f(x) = \log_{10} x = \log x$.

EXAMPLE 3: Evaluate $\lim_{x \to 1} \log x$.

Solution. We will construct the table of values for $f(x) = \log x$. We first approach the number 1 from the left or through the values less than but close to 1.

x	$f(x)$
0.1	-1
0.5	-0.30102999566
0.9	-0.04575749056
0.99	-0.0043648054
0.999	-0.00043451177
0.9999	-0.00004343161
0.99999	-0.00000434296

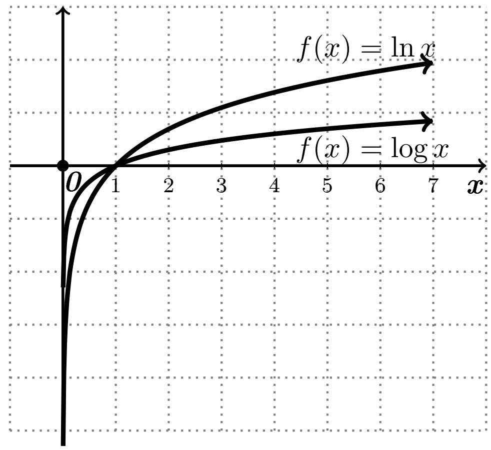
Now we consider approaching 1 from its right or through values greater than but close to 1.

x	$f(x)$
2	0.30102999566
1.5	0.17609125905
1.1	0.04139268515
1.01	0.00432137378
1.001	0.00043407747
1.0001	0.00004342727
1.00001	0.00000434292

As the values of x get closer and closer to 1, the values of $f(x)$ get closer and closer to 0. In symbols,

$$\lim_{x \to 1} \log x = 0.$$

Consider now the graphs of both the natural and common logarithmic functions. We can use the following graphs to determine their limits as x approaches 1..



The figure helps verify our observations that $\lim_{x\to 1} \ln x = 0$ and $\lim_{x\to 1} \log x = 0$. Also, based on the figure, we have

(a) $\lim_{x\to e} \ln x = 1$

(b) $\lim_{x\to 10} \log x = 1$

(c) $\lim_{x\to 3} \ln x = \ln 3 = 1.09...$

(d) $\lim_{x\to 3} \log x = \log 3 = 0.47...$

(e) $\lim_{x\to 0^+} \ln x = -\infty$

(f) $\lim_{x\to 0^+} \log x = -\infty$

TRIGONOMETRIC FUNCTIONS

EXAMPLE 4: Evaluate $\lim_{x\to 0} \sin x$.

Solution. We will construct the table of values for $f(x) = \sin x$. We first approach 0 from the left or through the values less than but close to 0.

x	$f(x)$
-1	-0.8414709848
-0.5	-0.4794255386
-0.1	-0.09983341664
-0.01	-0.00999983333
-0.001	-0.00099999983
-0.0001	-0.00009999999
-0.00001	-0.00000999999

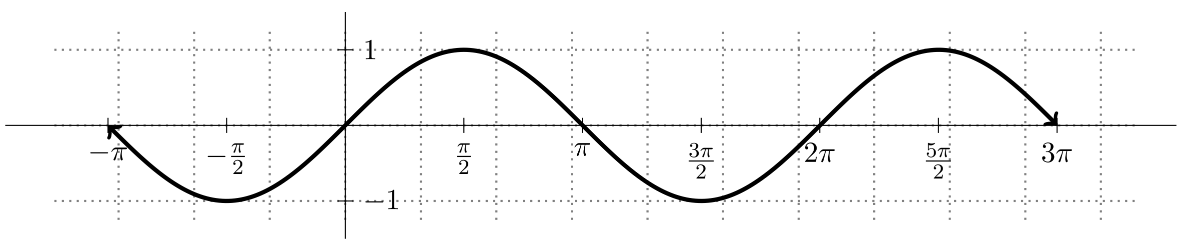
Now we consider approaching 0 from its right or through values greater than but close to 0.

x	$f(x)$
1	0.8414709848
0.5	0.4794255386
0.1	0.09983341664
0.01	0.00999983333
0.001	0.00099999983
0.0001	0.00009999999
0.00001	0.00000999999

As the values of x get closer and closer to 1, the values of $f(x)$ get closer and closer to 0. In symbols,

$$\lim_{x \to 0} \sin x = 0.$$

We can also find $\lim_{x \to 0} \sin x$ by using the graph of the sine function. Consider the graph of $f(x) = \sin x$.



The graph validates our observation in Example 4 that $\lim_{x \to 0} \sin x = 0$. Also, using the graph, we have the following:

(a) $\lim_{x \to \frac{\pi}{2}} \sin x = 1$.

(b) $\lim_{x \to \pi} \sin x = 0$.

(c) $\lim_{x \to -\frac{\pi}{2}} \sin x = -1$.

(d) $\lim_{x \to -\pi} \sin x = 0$.

Teaching Tip

Ask the students what they have observed about the limit of the functions above and their functional value at a point. Lead them to the fact that if f is either exponential, logarithmic or trigonometric, and if c is a real number which is in the domain of f, then

$$\lim_{x \to c} f(x) = f(c).$$

This property is also shared by polynomials and rational functions, as discussed in Topic 1.4.

(C) EXERCISES

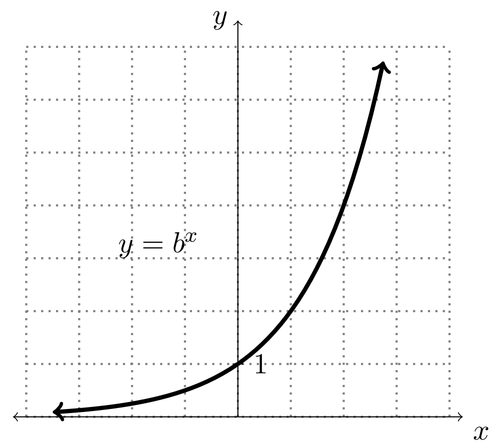
I. Evaluate the following limits by constructing the table of values.

1. $\lim_{x \to 1} 3^x$
2. $\lim_{x \to 2} 5^x$
3. $\lim_{x \to 4} \log x$
4. $\lim_{x \to 0} \cos x$
5. $\lim_{x \to 0} \tan x$

$\star$6. $\lim_{x \to \pi} \cos x$ Answer: -1

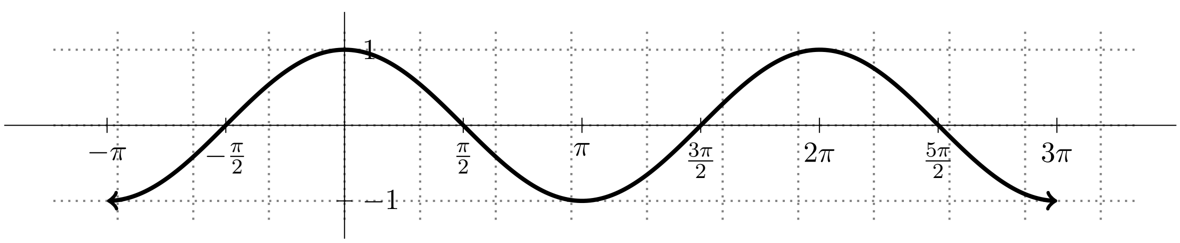
$\star$7. $\lim_{x \to \pi} \sin x$ Answer: 0

II. Given the graph below, evaluate the following limits:



1. $\lim_{x \to 0} b^x$
2. $\lim_{x \to 1.2} b^x$
3. $\lim_{x \to -1} b^x$

III. Given the graph of the cosine function $f(x) = \cos x$, evaluate the following limits:



1. $\lim_{x \to 0} \cos x$
2. $\lim_{x \to \pi} \cos x$
3. $\lim_{x \to \frac{\pi}{2}} \cos x$

TOPIC 2.2: Some Special Limits

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

We will determine the limits of three special functions; namely, $f(t) = \dfrac{\sin t}{t}$, $g(t) = \dfrac{1-\cos t}{t}$, and $h(t) = \dfrac{e^t - 1}{t}$. These functions will be vital to the computation of the derivatives of the sine, cosine, and natural exponential functions in Chapter 2.

(B) LESSON PROPER

THREE SPECIAL FUNCTIONS

We start by evaluating the function $f(t) = \dfrac{\sin t}{t}$.

EXAMPLE 1: Evaluate $\displaystyle\lim_{t\to 0} \frac{\sin t}{t}$.

Solution. We will construct the table of values for $f(t) = \dfrac{\sin t}{t}$. We first approach the number 0 from the left or through values less than but close to 0.

t	$f(t)$
-1	0.84147099848
-0.5	0.9588510772
-0.1	0.9983341665
-0.01	0.9999833334
-0.001	0.9999998333
-0.0001	0.99999999983

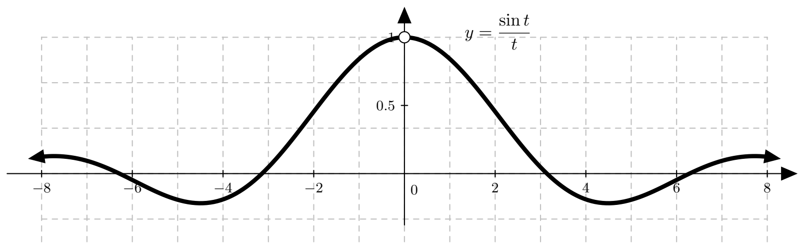
Now we consider approaching 0 from the right or through values greater than but close to 0.

t	$f(t)$
1	0.8414709848
0.5	0.9588510772
0.1	0.9983341665
0.01	0.9999833334
0.001	0.9999998333
0.0001	0.9999999983

Since $\lim_{t \to 0^-} \frac{\sin t}{t}$ and $\lim_{t \to 0^+} \frac{\sin t}{t}$ are both equal to 1, we conclude that

$$\lim_{t \to 0} \frac{\sin t}{t} = 1.$$

The graph of $f(t) = \frac{\sin t}{t}$ below confirms that the y-values approach 1 as t approaches 0.



Now, consider the function $g(t) = \frac{1 - \cos t}{t}$.

EXAMPLE 2: Evaluate $\lim_{t \to 0} \frac{1 - \cos t}{t}$.

Solution. We will construct the table of values for $g(t) = \frac{1 - \cos t}{t}$. We first approach the number 1 from the left or through the values less than but close to 0.

t	$g(t)$
-1	-0.4596976941
-0.5	-0.2448348762
-0.1	-0.04995834722
-0.01	-0.0049999583
-0.001	-0.0004999999
-0.0001	-0.000005

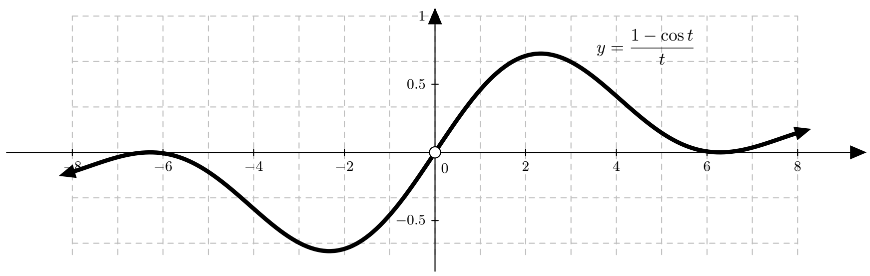
Now we consider approaching 0 from the right or through values greater than but close to 0.

t	$g(t)$
1	0.4596976941
0.5	0.2448348762
0.1	0.04995834722
0.01	0.0049999583
0.001	0.0004999999
0.0001	0.000005

Since $\lim_{t\to 0^-} \frac{1-\cos t}{t} = 0$ and $\lim_{t\to 0^+} \frac{1-\cos t}{t} = 0$, we conclude that

$$\lim_{t\to 0} \frac{1-\cos t}{t} = 0.$$

Below is the graph of $g(t) = \frac{1-\cos t}{t}$. We see that the y-values approach 0 as t tends to 0.



We now consider the special function $h(t) = \frac{e^t - 1}{t}$.

EXAMPLE 3: Evaluate $\lim_{t\to 0} \frac{e^t - 1}{t}$.

Solution. We will construct the table of values for $h(t) = \frac{e^t - 1}{t}$. We first approach the number 0 from the left or through the values less than but close to 0.

t	$h(t)$
-1	0.6321205588
-0.5	0.7869386806
-0.1	0.9516258196
-0.01	0.9950166251
-0.001	0.9995001666
-0.0001	0.9999500016

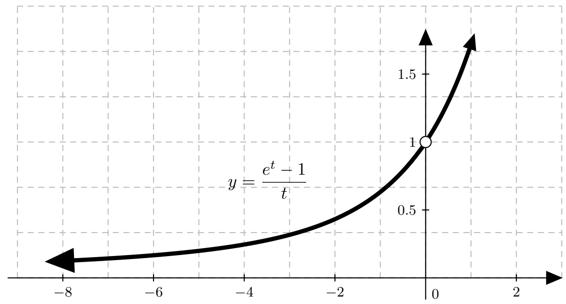
Now we consider approaching 0 from the right or through values greater than but close to 0.

t	$h(t)$
1	1.718281828
0.5	1.297442541
0.1	1.051709181
0.01	1.005016708
0.001	1.000500167
0.0001	1.000050002

Since $\lim_{x \to 0^-} \dfrac{e^t - 1}{t} = 1$ and $\lim_{x \to 0^+} \dfrac{e^t - 1}{t} = 1$, we conclude that

$$\lim_{x \to 0} \frac{e^t - 1}{t} = 1.$$

The graph of $h(t) = \dfrac{e^t - 1}{t}$ below confirms that $\lim_{t \to 0} h(t) = 1$.



INDETERMINATE FORM $\dfrac{\text{“}0\text{”}}{0}$

There are functions whose limits cannot be determined immediately using the Limit Theorems we have so far. In these cases, the functions must be manipulated so that the limit, if it exists, can be calculated. We call such limit expressions *indeterminate forms*.

In this lesson, we will define a particular indeterminate form, $\dfrac{\text{“}0\text{”}}{0}$, and discuss how to evaluate a limit which will initially result in this form.

Definition of Indeterminate Form of Type $\dfrac{\text{“}0\text{”}}{0}$

If $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} \dfrac{f(x)}{g(x)}$ is called an **indeterminate form of type** $\dfrac{\text{“}0\text{”}}{0}$.

Remark 1: A limit that is indeterminate of type $\dfrac{\text{“}0\text{”}}{0}$ may exist. To find the actual value, one should find an expression equivalent to the original. This is commonly done by factoring or by rationalizing. Hopefully, the expression that will emerge after factoring or rationalizing will have a computable limit.

EXAMPLE 4: Evaluate $\lim_{x \to -1} \dfrac{x^2 + 2x + 1}{x + 1}$.

Solution. The limit of both the numerator and the denominator as x approaches -1 is 0. Thus, this limit as currently written is an indeterminate form of type $\dfrac{0}{0}$. However, observe that $(x + 1)$ is a factor common to the numerator and the denominator, and

$$\frac{x^2 + 2x + 1}{x + 1} = \frac{(x + 1)^2}{x + 1} = x + 1, \text{ when } x \neq -1.$$

Therefore,

$$\lim_{x\to -1}\frac{x^2+2x+1}{x+1}=\lim_{x\to -1}(x+1)=0.$$

EXAMPLE 5: Evaluate $\displaystyle\lim_{x\to 1}\frac{x^2-1}{\sqrt{x}-1}$.

Solution. Since $\displaystyle\lim_{x\to 1}x^2-1=0$ and $\displaystyle\lim_{x\to 1}\sqrt{x}-1=0$, then $\displaystyle\lim_{x\to 1}\frac{x^2-1}{\sqrt{x}-1}$ is an indeterminate form of type $\displaystyle\text{“}\frac{0}{0}\text{”}$. To find the limit, observe that if $x\neq 1$, then

$$\frac{x^2-1}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=\frac{(x-1)(x+1)(\sqrt{x}+1)}{x-1}=(x+1)(\sqrt{x}+1).$$

So, we have

$$\lim_{x\to 1}\frac{x^2-1}{\sqrt{x}+1}=\lim_{x\to 1}(x+1)(\sqrt{x}+1)=4.$$

Teaching Tip

In solutions of evaluating limits, it is a common mistake among students to forget to write the “*lim*" operator. They will write

$$\lim_{x\to 1}\frac{x^2-1}{\sqrt{x}+1}=(x+1)(\sqrt{x}+1)=4,$$

instead of always writing the limit operator until such time that they are already substituting the value $x=1$. Of course, mathematically, the equation above does not make sense since $(x+1)(\sqrt{x}+1)$ is not always equal to 4. Please stress the importance of the “*lim*" operator.

Remark 2: We note here that the three limits discussed in Part 1 of this section,

$$\lim_{t\to 0}\frac{\sin t}{t},\quad \lim_{t\to 0}\frac{1-\cos t}{t},\quad \text{and}\quad \lim_{x\to 0}\frac{e^t-1}{t},$$

will result in $\displaystyle\text{“}\frac{0}{0}\text{”}$ upon direct substitution. However, they are not resolved by factoring or rationalization, but by a method which you will learn in college calculus.

(C) **EXERCISES**

I. Evaluate the following limits by constructing their respective tables of values.

1. $\lim_{t\to 0} \frac{t}{\sin t}$
2. $\lim_{t\to 0} \frac{t}{e^t - 1}$
3. $\lim_{t\to 0} \frac{\sin(2t)}{2t}$
4. $\lim_{t\to 0} \frac{1-\cos(3t)}{3t}$

$\star$5. $\lim_{t\to 0} \frac{\sin t}{t} \cdot \frac{1-\cos t}{t}$ Answer: 0

$\star$6. $\lim_{t\to 0} \frac{1-\cos t}{\sin t}$ Answer: 0

II. Evaluate the following limits:

1. $\lim_{w\to 1} (1+\sqrt[3]{w})(2-w^2+3w^3)$
2. $\lim_{t\to -1} \frac{t^2-1}{t^2+4t+3}$
3. $\lim_{z\to 2} \left(\frac{2z-z^2}{z^2-4}\right)^3$
4. $\lim_{x\to -1} \frac{x^2-x-2}{x^3-6x^2-7x}$
5. $\lim_{y\to -2} \frac{4-3y^2-y^3}{6-y-2y^2}$
6. $\lim_{x\to 4} \frac{x^3-7x^2+14x-8}{x^2-3x-4}$
7. $\lim_{x\to -1} \frac{\sqrt{x^2+3}-2}{x^2-1}$
8. $\lim_{x\to 2} \frac{\sqrt{2x}-\sqrt{6-x}}{4-x^2}$

$\star$9. $\lim_{x\to 16} \frac{x^2-256}{4-\sqrt{x}}$ Answer: -256

$\star$10. $\lim_{q\to -1} \frac{\sqrt{9q^2-4}-\sqrt{17+12q}}{q^2+3q+2}$ Answer: $-3\sqrt{5}$

LESSON 3: Continuity of Functions

TIME FRAME: 3-4 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate continuity of a function at a point;
2. Determine whether a function is continuous at a point or not;
3. Illustrate continuity of a function on an interval; and
4. Determine whether a function is continuous on an interval or not.

LESSON OUTLINE:

1. Continuity at a point
2. Determining whether a function is continuous or not at a point
3. Continuity on an interval
4. Determining whether a function is continuous or not on an interval

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TOPIC 3.1: Continuity at a Point

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

As we have observed in our discussion of limits in Topic (1.2), there are functions whose limits are not equal to the function value at $x = c$, meaning, $\lim_{x \to c} f(x) \neq f(c)$.

$\lim_{x \to c} f(x)$ is NOT NECESSARILY the same as $f(c)$.

This leads us to the study of continuity of functions. In this section, we will be focusing on the continuity of a function at a specific point.

Teaching Tip

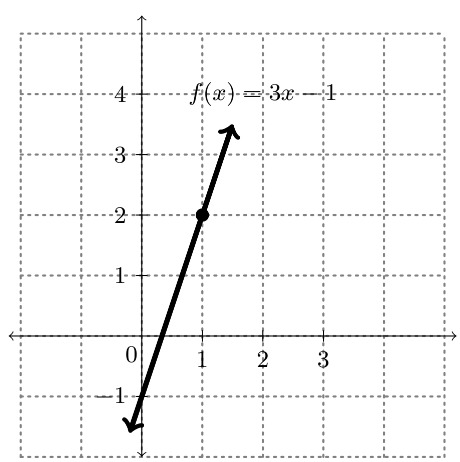
Ask the students to describe, in their own words, the term *continuous*. Ask them how the graph of a continuous function should look. Lead them towards the conclusion that a graph describes a continuous function if they can draw the entire graph without lifting their pen, or pencil, from their sheet of paper.

(B) LESSON PROPER

LIMITS AND CONTINUITY AT A POINT

What does "continuity at a point" mean? Intuitively, this means that in drawing the graph of a function, the point in question will be traversed. We start by graphically illustrating what it means to be continuity at a point.

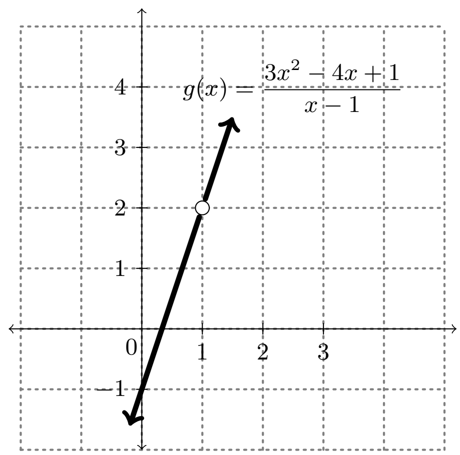
EXAMPLE 1: Consider the graph below.



Is the function continuous at $x = 1$?

Solution. To check if the function is continuous at $x = 1$, use the given graph. Note that one is able to trace the graph from the left side of the number $x = 1$ going to the right side of $x = 1$, without lifting one's pen. This is the case here. Hence, we can say that the function is continuous at $x = 1$.

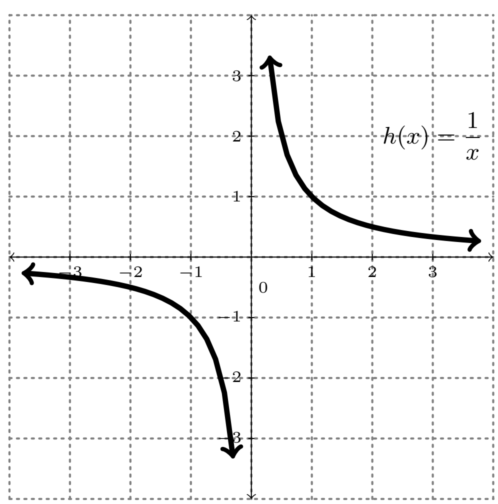
EXAMPLE 2: Consider the graph of the function $g(x)$ below.



Is the function continuous at $x = 1$?

Solution. We follow the process in the previous example. Tracing the graph from the left of $x = 1$ going to right of $x = 1$, one finds that s/he must lift her/his pen briefly upon reaching $x = 1$, creating a hole in the graph. Thus, the function is discontinuous at $x = 1$.

EXAMPLE 3: Consider the graph of the function $h(x) = \dfrac{1}{x}$.



Is the function continuous at $x = 0$?

Solution. If we trace the graph from the left of $x = 0$ going to right of $x = 0$, we have to lift our pen since at the left of $x = 0$, the function values will go downward indefinitely, while at the right of $x = 0$, the function values will go to upward indefinitely. In other words,

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \text{ and } \lim_{x\to 0^+} \frac{1}{x} = \infty$$

Thus, the function is discontinuous at $x = 0$.

EXAMPLE 4: Consider again the graph of the function $h(x) = \dfrac{1}{x}$. Is the function continuous at $x = 2$?

Solution. If we trace the graph of the function $h(x) = \dfrac{1}{x}$ from the left of $x = 2$ to the right of $x = 2$, you will not lift your pen. Therefore, the function h is continuous at $x = 2$.

Suppose we are not given the graph of a function but just the function itself. How do we determine if the function is continuous at a given number? In this case, we have to check three conditions.

Three Conditions of Continuity

A function $f(x)$ is said to be **continuous** at $x = c$ if the following three conditions are satisfied:

(i) $f(c)$ exists;

(ii) $\lim_{x\to c} f(x)$ exists; and

(iii) $f(c) = \lim_{x\to c} f(x)$.

If at least one of these conditions is not met, f is said to be **discontinuous** at $x = c$.

EXAMPLE 5: Determine if $f(x) = x^3 + x^2 - 2$ is continuous or not at $x = 1$.

Solution. We have to check the three conditions for continuity of a function.

(a) If $x = 1$, then $f(1) = 0$.

(b) $\lim_{x\to 1} f(x) = \lim_{x\to 1}(x^3 + x^2 - 2) = 1^3 + 1^2 - 2 = 0$.

(c) $f(1) = 0 = \lim_{x\to 1} f(x)$.

Therefore, f is continuous at $x = 1$.

EXAMPLE 6: Determine if $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ is continuous or not at $x = 0$.

Solution. We have to check the three conditions for continuity of a function.

(a) If $x = 0$, then $f(0) = 1$.

(b) $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{x^2 - x - 2}{x - 2} = \lim\limits_{x \to 0} \dfrac{(x-2)(x+1)}{x - 2} = \lim\limits_{x \to 0} (x + 1) = 1$.

(c) $f(0) = 1 = \lim\limits_{x \to 0} f(x)$.

Therefore, f is continuous at $x = 0$.

EXAMPLE 7: Determine if $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ is continuous or not at $x = 2$.

Solution. Note that f is not defined at $x = 2$ since 2 is not in the domain of f. Hence, the first condition in the definition of a continuous function is not satisfied. Therefore, f is discontinuous at $x = 2$.

EXAMPLE 8: Determine if

$$f(x) = \begin{cases} x + 1 & \text{if } x < 4, \\ (x-4)^2 + 3 & \text{if } x \geq 4 \end{cases}$$

is continuous or not at $x = 4$. (This example was given in Topic 1.1.)

Solution. Note that f is defined at $x = 4$ since $f(4) = 3$. However, $\lim\limits_{x \to 4^-} f(x) = 5$ while $\lim\limits_{x \to 4^+} f(x) = 3$. Therefore $\lim\limits_{x \to 4^-} f(x)$ DNE, and f is discontinuous at $x = 4$.

Teaching Tip

The following seatwork is suggested at this point: Determine if $f(x) = \sqrt{x - 1}$ is continuous or not at $x = 4$.

Solution. We check the three conditions:

(a) $f(4) = \sqrt{4 - 1} = \sqrt{3} > 0$

(b) $\lim\limits_{x \to 4} \sqrt{x - 1} = \sqrt{4 - 1} = \sqrt{3}$

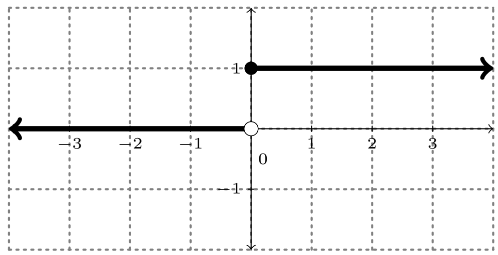
(c) $f(4) = \sqrt{3} = \lim\limits_{x \to 4} \sqrt{x - 1}$

Therefore, the function f is continuous at $x = 4$.

(C) **EXERCISES**

I. Given the graph below, determine if the function $H(x)$ is continuous at the following values of x:

1. $x = 2$
2. $x = -3$
3. $x = 0$



Heaviside function $H(x)$

II. Determine if the following functions are continuous at the given value of x.

1. $f(x) = 3x^2 + 2x + 1$ at $x = -2$
2. $f(x) = 9x^2 - 1$ at $x = 1$
3. $f(x) = \dfrac{1}{x-2}$ at $x = 2$
4. $h(x) = \dfrac{x-1}{x^2-1}$ at $x = 1$
5. $h(x) = \dfrac{x+1}{x^2-1}$ at $x = 1$
6. $g(x) = \sqrt{x-3}$ at $x = 4$
7. $g(x) = \dfrac{x}{\sqrt{4-x}}$ at $x = 8$
8. $g(x) = \dfrac{\sqrt{4-x}}{x}$ at $x = 0$

TOPIC 3.2: Continuity on an Interval

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

A function can be continuous on an interval. This simply means that it is continuous at every point on the interval. Equivalently, if we are able to draw the entire graph of the function on an interval without lifting our tracing pen, or without being interrupted by a hole in the middle of the graph, then we can conclude that the the function is continuous on that interval.

We begin our discussion with two concepts which are important in determining whether a function is continuous at the endpoints of closed intervals.

One-Sided Continuity

(a) A function f is said to be **continuous from the left at** $x = c$ if

$$f(c) = \lim_{x \to c^-} f(x).$$

(b) A function f is said to be **continuous from the right at** $x = c$ if

$$f(c) = \lim_{x \to c^+} f(x).$$

Here are known facts on continuities of functions on intervals:

Continuity of Polynomial, Absolute Value, Rational and Square Root Functions

(a) Polynomial functions are continuous everywhere.

(b) The absolute value function $f(x) = |x|$ is continuous everywhere.

(c) Rational functions are continuous on their respective domains.

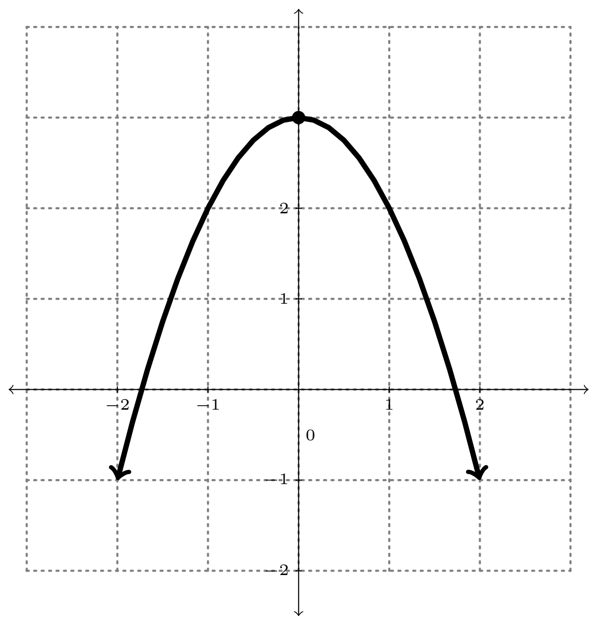
(d) The square root function $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

(B) **LESSON PROPER**

LIMITS AND CONTINUITY ON AN INTERVAL

We first look at graphs of functions to illustrate continuity on an interval.

EXAMPLE 1: Consider the graph of the function f given below.



Using the given graph, determine if the function f is continuous on the following intervals:

(a) $(-1, 1)$ (b) $(-\infty, 0)$ (c) $(0, +\infty)$

Solution. Remember that when we say "trace from the right side of $x = c$", we are tracing not from $x = c$ on the x-axis, but from the point $(c, f(c))$ *along the graph.*

(a) We can trace the graph from the right side of $x = -1$ to the left side of $x = 1$ without lifting the pen we are using. Hence, we can say that the function f is continuous on the interval $(-1, 1)$.

(b) If we trace the graph from any negatively large number up to the left side of 0, we will not lift our pen and so, f is continuous on $(-\infty, 0)$.

(c) For the interval $(0, +\infty)$, we trace the graph from the right side of 0 to any large number, and find that we will not lift our pen. Thus, the function f is continuous on $(0, +\infty)$.

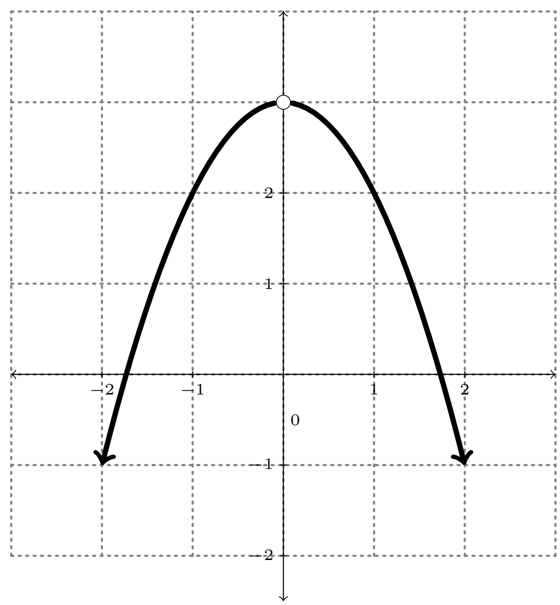
Teaching Tip

Please point these out after solving the previous example:

(a) The function is actually continuous on $[-1, 1]$, $[0, +\infty)$ and $(-\infty, 0]$ since the function f is defined at the endpoints of the intervals: $x = -1$, $x = 1$, and $x = 0$, and we are still able to trace the graph on these intervals without lifting our tracing pen.

(b) The function f is therefore continuous on the interval $(-\infty, +\infty)$ since if we trace the entire graph from left to right, we won't be lifting our pen. This is an example of a function which is continuous everywhere.

EXAMPLE 2: Consider the graph of the function h below.



Determine using the given graph if the function f is continuous on the following intervals:

a. $(-1, 1)$

b. $[0.5, 2]$

Solution. Because we are already given the graph of h, we characterize the continuity of h by the possibility of tracing the graph without lifting the pen.

(a) If we trace the graph of the function h from the right side of $x = -1$ to the left side of $x = 1$, we will be interrupted by a hole when we reach $x = 0$. We are forced to lift our pen just before we reach $x = 0$ to indicate that h is not defined at $x = 0$ and continue tracing again starting from the right of $x = 0$. Therefore, we are not able to trace the graph of h on $(-1, 1)$ without lifting our pen. Thus, the function h is not continuous on $(-1, 1)$.

(b) For the interval $[0.5, 2]$, if we trace the graph from $x = 0.5$ to $x = 2$, we do not have to lift the pen at all. Thus, the function h is continuous on $[0.5, 2]$.

Now, if a function is given without its corresponding graph, we must find other means to determine if the function is continuous or not on an interval. Here are definitions that will help us:

A function f is said to be **continuous**...

(a) everywhere if f is continuous at every real number. In this case, we also say f is continuous on $\mathbb{R}$.

(b) on (a, b) if f is continuous at every point x in (a, b).

(c) on $[a, b)$ if f is continuous on (a, b) and from the right at a.

(d) on $(a, b]$ if f is continuous on (a, b) and from the left at b.

(e) on $[a, b]$ if f is continuous on $(a, b]$ and on $[a, b)$.

(f) on (a, ∞) if f is continuous at all $x > a$.

(g) on $[a, \infty)$ if f is continuous on (a, ∞) and from the right at a.

(h) on $(-\infty, b)$ if f is continuous at all $x < b$.

(i) on $(-\infty, b]$ if f is continuous on $(-\infty, b)$ and from the left at b.

EXAMPLE 3: Determine the largest interval over which the function $f(x) = \sqrt{x+2}$ is continuous.

Solution. Observe that the function $f(x) = \sqrt{x+2}$ has function values only if $x + 2 \geq 0$, that is, if $x \in [-2, +\infty)$. For all $c \in (-2, +\infty)$,

$$f(c) = \sqrt{c+2} = \lim_{x \to c} \sqrt{x+2}.$$

Moreover, f is continuous from the right at -2 because

$$f(-2) = 0 = \lim_{x \to -2^+} \sqrt{x+2}.$$

Therefore, for all $x \in [-2, +\infty)$, the function $f(x) = \sqrt{x+2}$ is continuous.

EXAMPLE 4: Determine the largest interval over which $h(x) = \dfrac{x}{x^2-1}$ is continuous.

Solution. Observe that the given rational function $h(x) = \dfrac{x}{x^2-1}$ is not defined at $x = 1$ and $x = -1$. Hence, the domain of h is the set $\mathbb{R} \setminus \{-1, 1\}$. As mentioned at the start of this topic, a rational function is continuous on its domain. Hence, h is continuous over $\mathbb{R} \setminus \{-1, 1\}$.

EXAMPLE 5: Consider the function $g(x) = \begin{cases} x & \text{if } x \leq 0, \\ 3 & \text{if } 0 < x \leq 1, \\ 3 - x^2 & \text{if } 1 < x \leq 4, \\ x - 3 & \text{if } x > 4. \end{cases}$

Is g continuous on $(0, 1]$? on $(4, \infty)$?

Solution. Since g is a piecewise function, we just look at the 'piece' of the function corresponding to the interval specified.

(a) On the interval $(0, 1]$, $g(x)$ takes the constant value 3. Also, for all $c \in (0, 1]$,

$$\lim_{x \to c} g(x) = 3 = g(c).$$

Thus, g is continuous on $(0, 1]$.

(b) For all $x > 4$, the corresponding 'piece' of g is $g(x) = x - 3$, a polynomial function. Recall that a polynomial function is continuous everywhere in $\mathbb{R}$. Hence, $f(x) = x - 3$ is surely continuous for all $x \in (4, +\infty)$.

(C) EXERCISES

1. Is the function $g(x) = \begin{cases} x & \text{if } x \leq 0, \\ 3 & \text{if } 0 < x \leq 1, \\ 3 - x^2 & \text{if } 1 < x \leq 4, \\ x - 3 & \text{if } x > 4, \end{cases}$ continuous on $[1, 4]$? on $(-\infty, 0)$?

$\star$2. Do as indicated.

a. Find all values of m such that $g(x) = \begin{cases} x + 1 & \text{if } x \leq m, \\ x^2 & \text{if } x > m, \end{cases}$ is continuous everywhere.

b. Find all values of a and b that make

$$h(x) = \begin{cases} x + 2a & \text{if } x < -2, \\ 3ax + b & \text{if } -2 \leq x \leq 1, \\ 3x - 2b & \text{if } x > 1, \end{cases}$$

continuous everywhere.

LESSON 4: More on Continuity

TIME FRAME: 4 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate different types of discontinuity (hole/removable, jump/essential, asymptotic/infinite);
2. Illustrate the Intermediate Value and Extreme Value Theorems; and
3. Solve problems involving the continuity of a function.

LESSON OUTLINE:

1. Review of continuity at a point
2. Illustration of a hole/removable discontinuity at a point
3. Illustration of a jump essential discontinuity at a point
4. Illustration of an infinite essential discontinuity at a point
5. Illustration of a consequence of continuity given by the Intermediate Value Theorem
6. Illustration of a consequence of continuity given by the Extreme Value Theorem
7. Situations which involve principles of continuity
8. Solutions to problems involving properties/consequences of continuity

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TOPIC 4.1: Different Types of Discontinuities

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

In Topic (1.2), it was emphasized that the value of $\lim_{x\to c} f(x)$ may be distinct from the value of the function itself at $x = c$. Recall that a limit may be evaluated at values which are not in the domain of $f(x)$.

In Topics (3.1) - (3.2), we learned that when $\lim_{x\to c} f(x)$ and $f(c)$ are equal, $f(x)$ is said to be continuous at c. Otherwise, it is said to be discontinuous at c. We will revisit the instances when $\lim_{x\to c} f(x)$ and $f(c)$ have unequal or different values. These instances of inequality and, therefore, discontinuity are very interesting to study. This section focuses on these instances.

(B) LESSON PROPER

Consider the functions $g(x)$, $h(x)$ and $j(x)$ where

$$g(x) = \begin{cases} \dfrac{3x^2 - 4x + 1}{x - 1} & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$$

$$h(x) = \begin{cases} x + 1 & \text{if } x < 4, \\ (x-4)^2 + 3 & \text{if } x \geq 4. \end{cases}$$

and

$$j(x) = \frac{1}{x}, \quad x \neq 0.$$

We examine these for continuity at the respective values 1, 4, and 0.

(a) $\lim_{x\to 1} g(x) = 2$ but $g(1) = 1$.

(b) $\lim_{x\to 4} h(x)$ DNE but $h(4) = 3$.

(c) $\lim_{x\to 0} j(x)$ DNE and $f(0)$ DNE.

All of the functions are discontinuous at the given values. A closer study shows that they actually exhibit different types of discontinuity.

REMOVABLE DISCONTINUITY

A function $f(x)$ is said to have a ***removable discontinuity*** at $x = c$ if

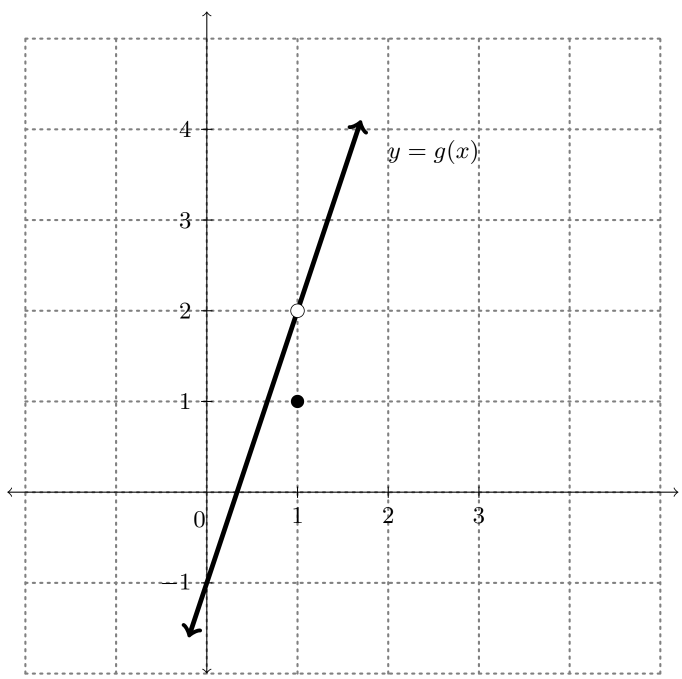
(a) $\lim_{x \to c} f(x)$ exists; and

(b) either $f(c)$ does not exist or $f(c) \neq \lim_{x \to c} f(x)$.

It is said to be *removable* because the discontinuity may be removed by *redefining* $f(c)$ so that it will equal $\lim_{x \to c} f(x)$. In other words, if $\lim_{x \to c} f(x) = L$, a removable discontinuity is remedied by the redefinition:

$$\text{Let } f(c) = L.$$

Recall $g(x)$ above and how it is discontinuous at 1. In this case, $g(1)$ exists. Its graph is as follows:



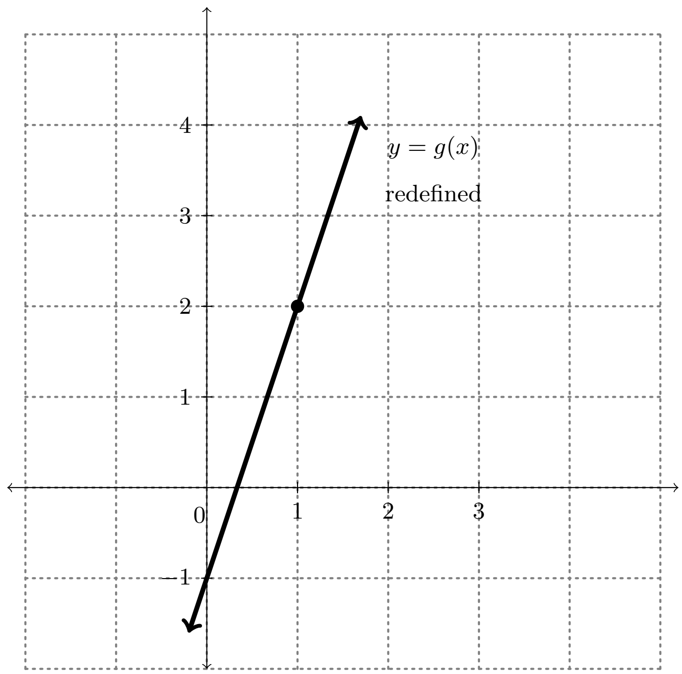
The discontinuity of g at the point $x = 1$ is manifested by the hole in the graph of $y = g(x)$ at the point $(1, 2)$. This is due to the fact that $f(1)$ is equal to 1 and not 2, while $\lim_{x \to 1} g(x) = 2$. We now demonstrate how this kind of a discontinuiy may be removed:

$$\text{Let } g(1) = 2.$$

This is called a *redefinition* of g at $x = 1$. The redefinition results in a "transfer" of the point $(1, 1)$ to the hole at $(1, 2)$. In effect, the hole is filled and the discontinuity is removed!

This is why the discontinuity is called a removable one. This is also why, sometimes, it is called a *hole* discontinuity.

We go back to the graph of $g(x)$ and see how redefining $f(1)$ to be 2 removes the discontinuity:



and revises the function to its continuous counterpart,

$$G(x) = \begin{cases} g(x) & \text{if } x \neq 1, \\ 2 & \text{if } x = 1. \end{cases}$$

ESSENTIAL DISCONTINUITY

A function $f(x)$ is said to have an **essential discontinuity** at $x = c$ if $\lim\limits_{x \to c} f(x)$ DNE.

<u>Case 1</u>. If for a function $f(x)$, $\lim\limits_{x \to c} f(x)$ DNE because the limits from the left and right of $x = c$ both exist but are not equal, that is,

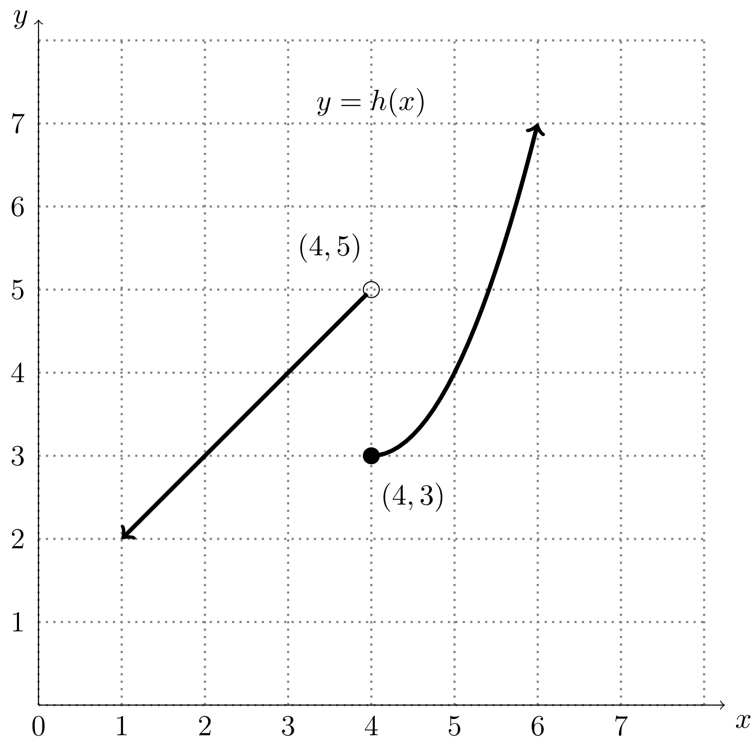
$$\lim_{x \to c^-} f(x) = L \text{ and } \lim_{x \to c^+} f(x) = M, \text{ where } L \neq M,$$

then f is said to have a *jump* essential discontinuity at $x = c$.

Recall the function $h(x)$ where

$$h(x) = \begin{cases} x+1 & \text{if } x < 4, \\ (x-4)^2+3 & \text{if } x \geq 4. \end{cases}$$

Its graph is as follows:



From Lesson 2, we know that $\lim_{x \to 4} h(x)$ DNE because

$$\lim_{x \to 4^-} h(x) = 5 \text{ and } \lim_{x \to 4^+} h(x) = 3.$$

The graph confirms that the discontinuity of $h(x)$ at $x = 4$ is certainly not removable. See, the discontinuity is not just a matter of having one point missing from the graph and putting it in; if ever, it is a matter of having a part of the graph entirely out of place. If we force to remove this kind of discontinuity, we need to connect the two parts by a vertical line from $(4, 5)$ to $(4, 3)$. However, the resulting graph will fail the Vertical Line Test and will not be a graph of a function anymore. Hence, this case has no remedy. From the graph, it is clear why this essential discontinuity is also called a *jump* discontinuity.

Case 2. If a function $f(x)$ is such that $\lim_{x \to c} f(x)$ DNE because either

(i) $\lim_{x \to c^-} f(x) = +\infty$, or

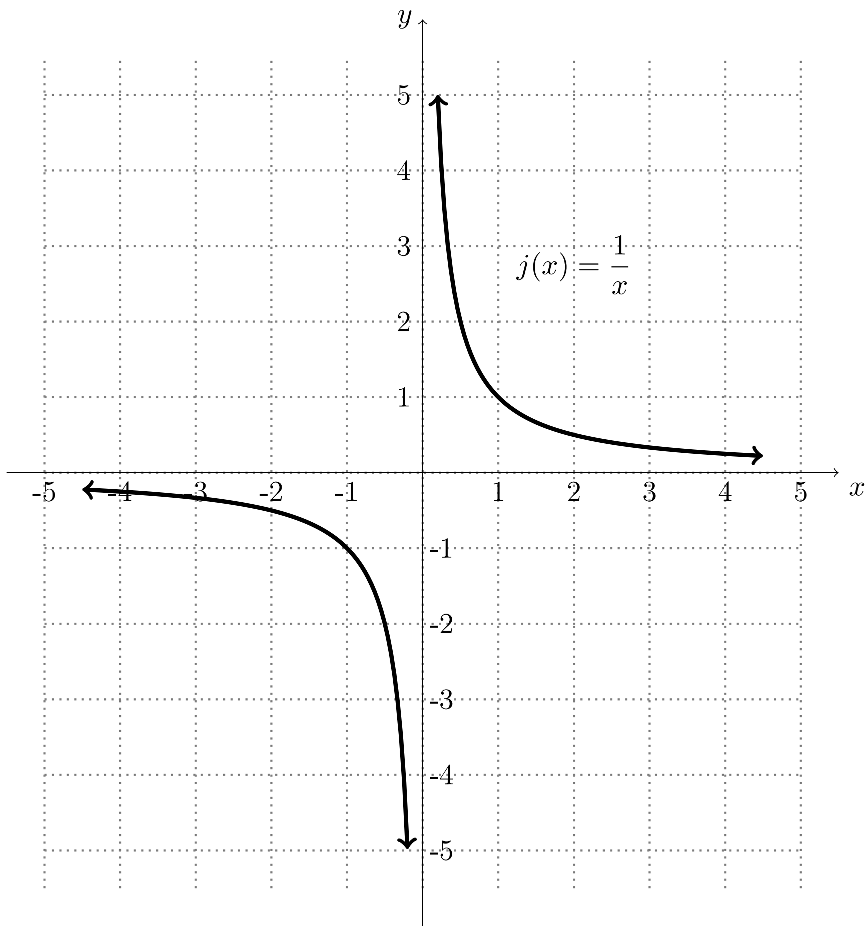
(ii) $\lim_{x \to c^-} f(x) = -\infty$, or

(iii) $\lim_{x \to c^+} f(x) = +\infty$, or

(iv) $\lim_{x \to c^+} f(x) = -\infty$,

then $f(x)$ is said to have an *infinite* discontinuity at $x = c$.

Recall $j(x) = \dfrac{1}{x}$, $x \neq 0$, as mentioned earlier. Its graph is as follows:

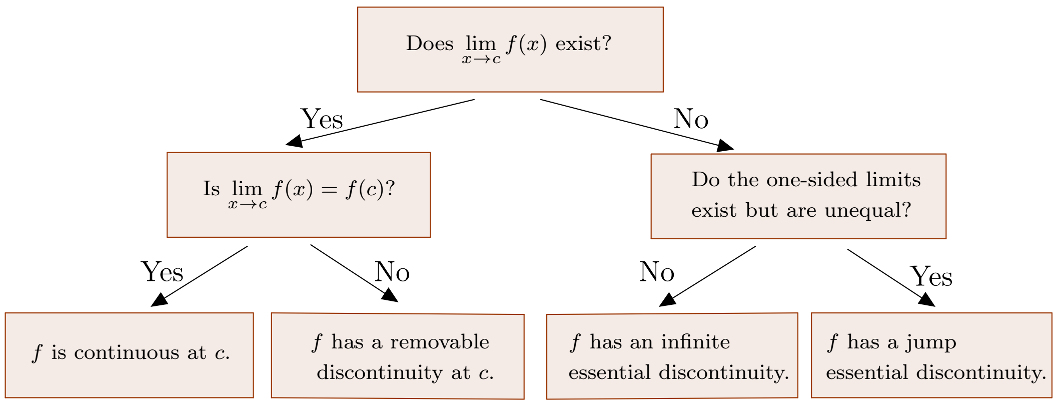


We have seen from Topic 1.4 that

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

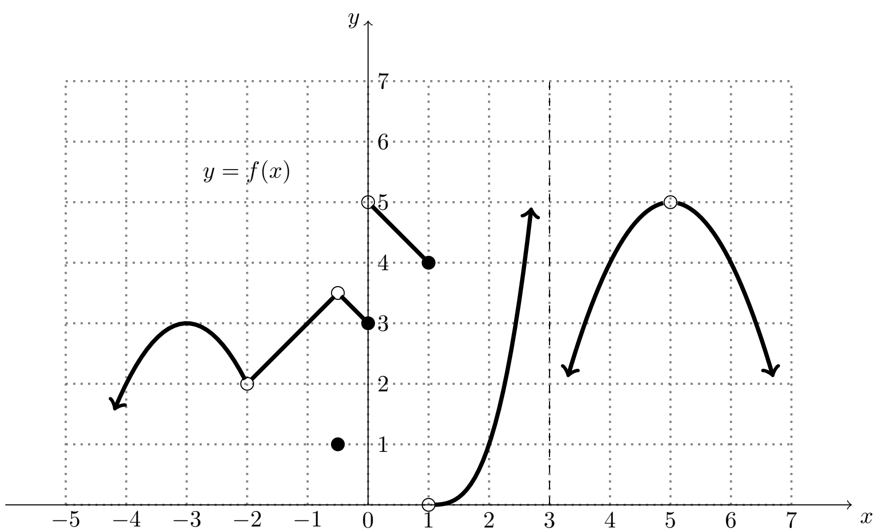
Because the limits are infinite, the limits from both the left and the right of $x = 0$ do not exist, and the discontinuity cannot be removed. Also, the absence of a left-hand (or right-hand) limit from which to "jump" to the other part of the graph means the discontinuity is permanent. As the graph indicates, the two ends of the function that approach $x = 0$ continuously move away from each other: one end goes upward without bound, the other end goes downward without bound. This translates to an asymptotic behavior as x-values approach 0; in fact, we say that $x = 0$ is a vertical asymptote of $f(x)$. Thus, this discontinuity is called an *infinite* essential discontinuity.

FLOWCHART. Here is a flowchart which can help evaluate whether a function is continuous or not at a point c. Before using this, make sure that the function is defined on an open interval containing c, except possibly at c.



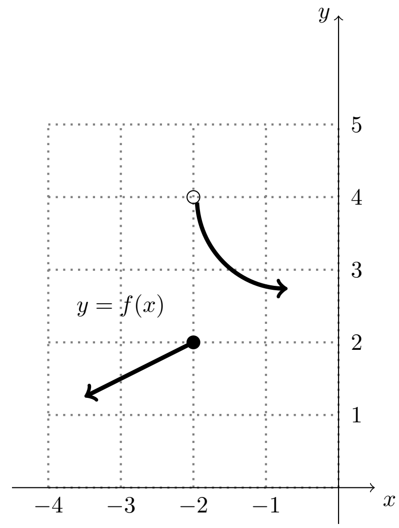
(C) **EXERCISES**

1. Consider the function $f(x)$ whose graph is given below.



Enumerate all discontinuities of $f(x)$ and identify their types. If a discontinuity is removable, state the redefiniton that will remove it. *Hint*: There are 6 discontinuities.

2. For each specified discontinuity, sketch the graph of a possible function $f(x)$ that illustrates the discontinuity. For example, if it has a jump discontinuity at $x = -2$, then a possible graph of f is



Do a similar rendition for f for each of the following discontinuities:

a. $\lim_{x\to 0} f(x) = 1$ and $f(0) = -3$

b. $\lim_{x\to 1} f(x) = -1$ and $f(1)$ DNE

c. $\lim_{x\to 2^-} f(x) = -2$ and $\lim_{x\to 2^+} f(x) = 2$

d. $\lim_{x\to 3^-} f(x) = -\infty$ and $\lim_{x\to 3^+} f(x) = +\infty$

e. $\lim_{x\to -1^-} f(x) = +\infty$, $\lim_{x\to -1^+} f(x) = 0$ and $f(-1) = 0$

f. $\lim_{x\to -1^-} f(x) = +\infty$, $\lim_{x\to -1^+} f(x) = 0$ and $f(-1) = -1$

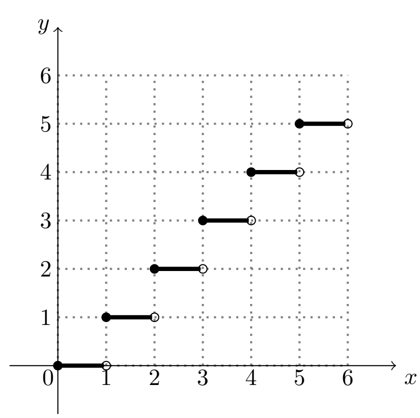
g. There is a removable discontinuity at $x = 1$ and $f(1) = 4$

h. There is a jump discontinuity at $x = 2$ and $f(2) = 3$

i. There is an infinite discontinuity at $x = 0$

j. There is an infinite discontinuity at $x = 0$ and $f(0) = -2$

3. Consider the function $f(x)$ whose graph is given below.



a. What kind of discontinuity is exhibited by the graph?

b. At what values of x does this type of discontinuity happen?

c. Can the discontinuities be removed? Why/Why not?

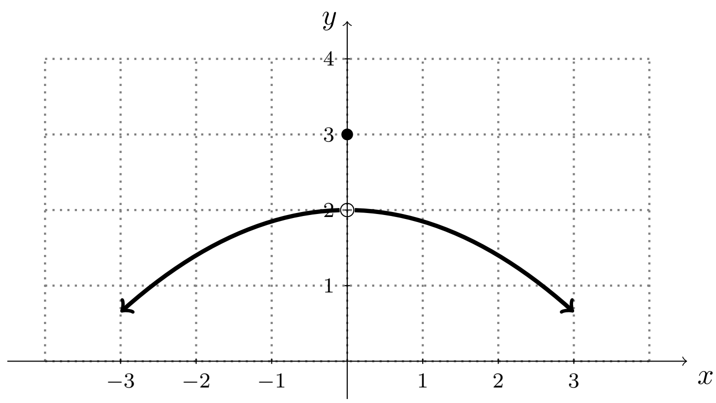
d. How many discontinuities do you see in the graph?

e. Based on the graph above, and assuming it is part of $f(x) = [\![x]\!]$, how many discontinuities will the graph of $f(x) = [\![x]\!]$ have?

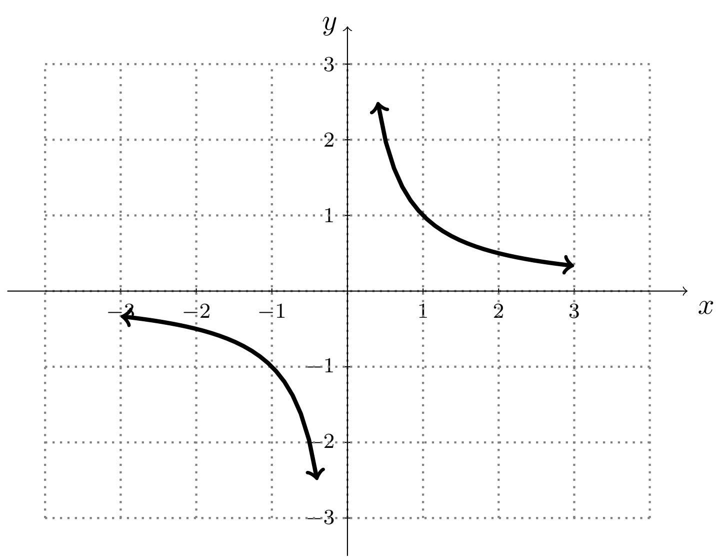
f. Assuming this is part of the graph of $f(x) = [\![x]\!]$, how would the discontinuities change if instead you have $f(x) = [\![2x]\!]$ or $f(x) = [\![3x]\!]$ or $f(x) = [\![0.5x]\!]$?

4. For each function whose graph is given below, identify the type(s) of discontinuity(ies) exhibited. Remedy any removable discontinuity with an appropriate redefinition.

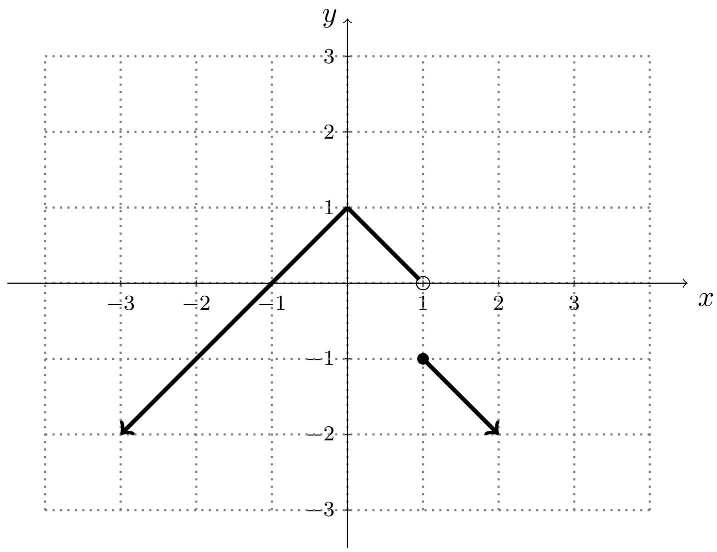
a. $y = f(x)$



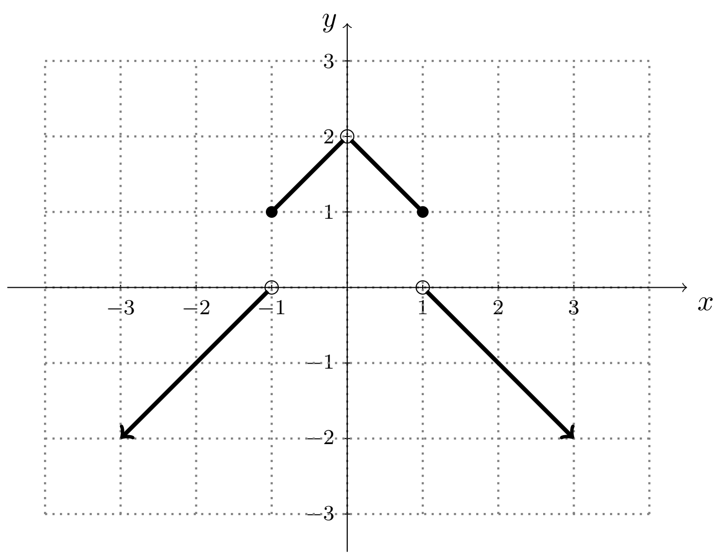
b. $y = g(x)$



c. $y = h(x)$



d. $y = j(x)$



5. Determine the possible points of discontinuity of the following functions and the type of discontinuity exhibited at that point. Remove any removable discontinuity. Sketch the graph of $f(x)$ to verify your answers.

a. $f(x) = \dfrac{1}{x^2}$

b. $f(x) = \dfrac{x^2 - 4}{x - 2}$

c. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x \neq 2, \\ -4 & \text{if } x = 2. \end{cases}$

d. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2, \\ -4 & \text{if } x \geq 2. \end{cases}$

e. $f(x) = \dfrac{x - 2}{x^2 - 4}$

f. $f(x) = \dfrac{1}{x^2 - 9}$

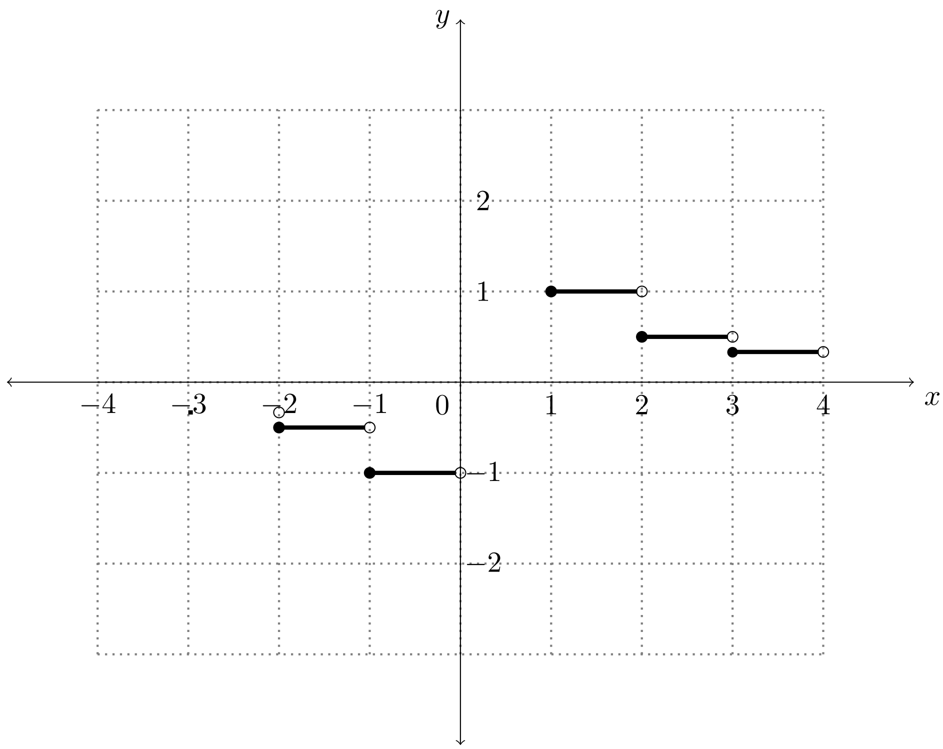
g. $f(x) = \tan x$

h. $f(x) = \cos x, \quad x \neq 2k\pi$, where k is an integer.

i. $f(x) = \csc x$

⋆j. $f(x) = \dfrac{1}{[\![x]\!]}$

Answer to the starred exercise: First of all, $f(x)$ will be discontinuous at values where the denominator will equal 0. This means that x cannot take values in the interval $[0, 1)$. This will cause a big jump (or essential) discontinuity from where the graph stops right before (-1,-1) to where it resumes at (1,1).
Moreover, there will again be jump discontinuities at the integer values of x.



TOPIC 4.2: The Intermediate Value and the Extreme Value Theorems

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

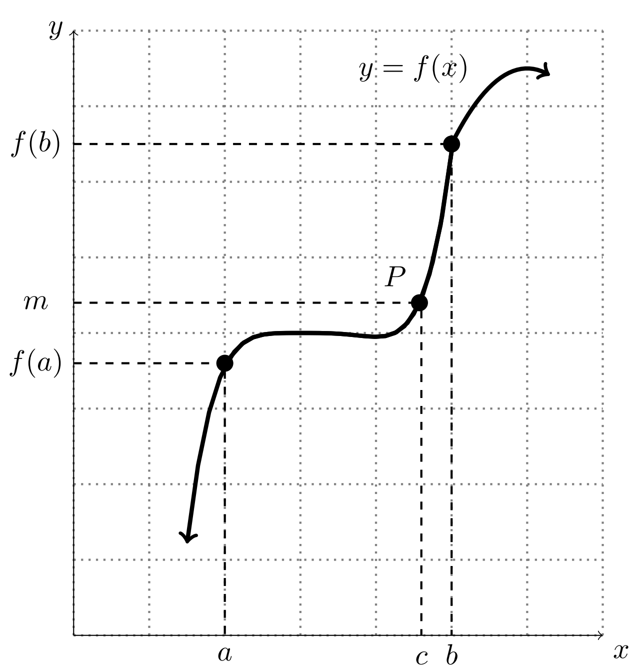
After discussing continuity at length, we will now learn two important consequences brought about by the continuity of a function over a closed interval. The first one is called the Intermediate Value Theorem or the IVT. The second one is called the Extreme Value Theorem or the EVT.

(B) LESSON PROPER

The Intermediate Value Theorem

The first theorem we will illustrate says that a function $f(x)$ which is found to be continuous over a closed interval $[a, b]$ will take any value between $f(a)$ and $f(b)$.

Theorem 4 (Intermediate Value Theorem (IVT))**.** *If a function $f(x)$ is continuous over a closed interval $[a, b]$, then for every value m between $f(a)$ and $f(b)$, there is a value $c \in [a, b]$ such that $f(c) = m$.*



Look at the graph as we consider values of m between $f(a)$ and $f(b)$. Imagine moving the dotted line for m up and down between the dotted lines for $f(a)$ and $f(b)$. Correspondingly, the dot P will move along the thickened curve between the two points, $(a, f(a))$ and $(b, f(b))$.

We make the following observations:

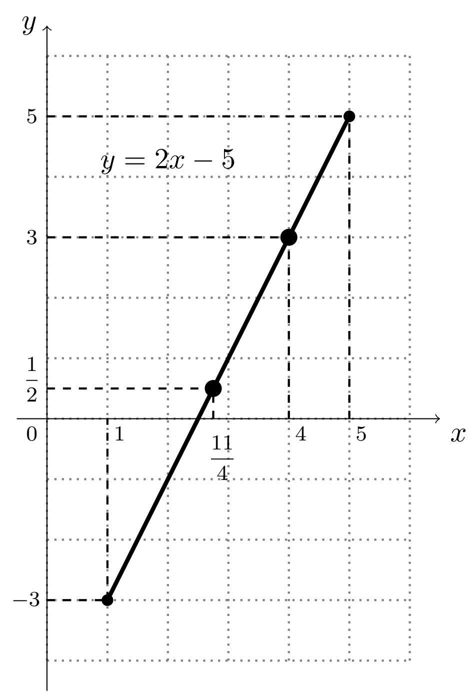
- As the dark dot moves, so will the vertical dotted line over $x = c$ move.
- In particular, the said line moves between the vertical dotted lines over $x = a$ and $x = b$.
- More in particular, for any value that we assign m in between $f(a)$ and $f(b)$, the consequent position of the dark dot assigns a corresponding value of c between a and b. This illustrates what the IVT says.

EXAMPLE 1: Consider the function $f(x) = 2x - 5$.

Since it is a linear function, we know it is continuous everywhere. Therefore, we can be sure that it will be continuous over any closed interval of our choice.

Take the interval $[1, 5]$. The IVT says that for any m intermediate to, or in between, $f(1)$ and $f(5)$, we can find a value intermediate to, or in between, 1 and 5.

Start with the fact that $f(1) = -3$ and $f(5) = 5$. Then, choose an $m \in [-3, 5]$, to exhibit a corresponding $c \in [1, 5]$ such that $f(c) = m$.



Choose $m = \dfrac{1}{2}$. By IVT, there is a $c \in [1, 5]$ such that $f(c) = \dfrac{1}{2}$. Therefore,

$$\frac{1}{2} = f(c) = 2c - 5 \quad \Longrightarrow \quad 2c = \frac{11}{2} \quad \Longrightarrow \quad c = \frac{11}{4}.$$

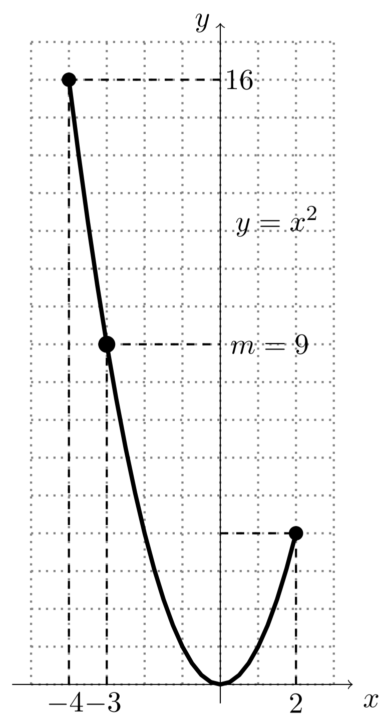
Indeed, $\dfrac{11}{4} \in (1, 5)$.

We can try another m-value in $(-3, 5)$. Choose $m = 3$. By IVT, there is a $c \in [1, 5]$ such that $f(c) = 3$. Therefore,

$$3 = f(c) = 2c - 5 \quad \Longrightarrow \quad 2c = 8 \quad \Longrightarrow \quad c = 4.$$

Again, the answer, 4, is in $[1, 5]$. The claim of IVT is clearly seen in the graph of $y = 2x - 5$.

EXAMPLE 2: Consider the simplest quadratic function $f(x) = x^2$.



Being a polynomial function, it is continuous everywhere. Thus, it is also continuous over any closed interval we may specify.

We choose the interval $[-4, 2]$. For any m in between $f(-4) = 16$ and $f(2) = 4$, there is a value c inside the interval $[-4, 2]$ such that $f(c) = m$.

Suppose we choose $m = 9 \in [4, 16]$. By IVT, there exists a number $c \in [-4, 2]$ such that $f(c) = 9$. Hence,

$$9 = f(c) = c^2 \quad \Longrightarrow \quad c = \pm 3.$$

However, we only choose $c = -3$ because the other solution $c = 3$ is not in the specified interval $[-4, 2]$.

Note: In the previous example, if the interval that was specified was [0,4], then the final answer would instead be $c = +3$.

Remark 1: The value of $c \in [a, b]$ in the conclusion of the Intermediate Value Theorem is not necessarily unique.

EXAMPLE 3: Consider the polynomial function

$$f(x) = x^3 - 4x^2 + x + 7$$

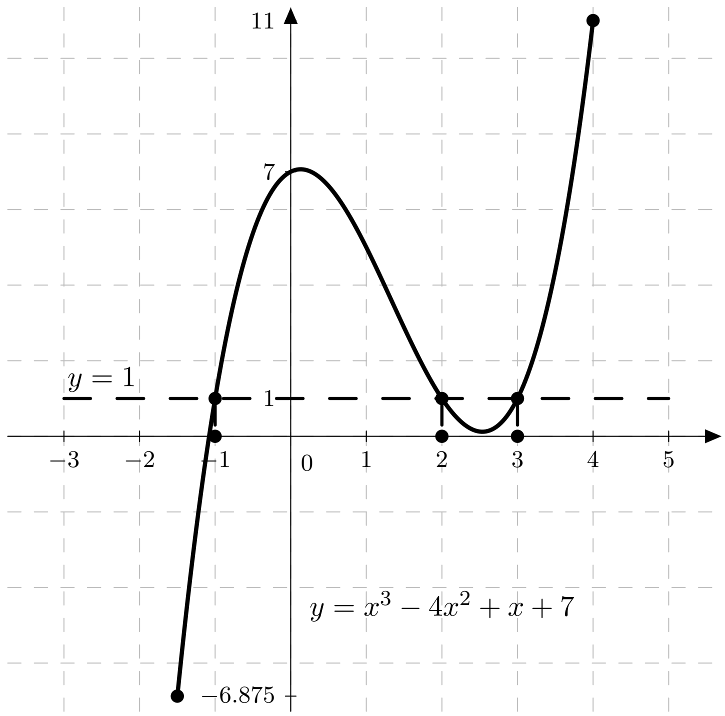
over the interval $[-1.5, 4]$ Note that

$$f(-1.5) = -6.875 \quad \text{and} \quad f(4) = 11.$$

We choose $m = 1$. By IVT, there exists $c \in [-1.5, 4]$ such that $f(c) = 1$. Thus,

$$\begin{aligned} & f(c) = c^3 - 4c^2 + c + 7 = 1 \\ \Longrightarrow \quad & c^3 - 4c^2 + c + 6 = 0 \\ \Longrightarrow \quad & (c+1)(c-2)(c-3) = 0 \\ \Longrightarrow \quad & c = -1 \text{ or } c = 2 \text{ or } c = 3. \end{aligned}$$

We see that there are three values of $c \in [-1.5, 4]$ which satisfy the conclusion of the Intermediate Value Theorem.



The Extreme Value Theorem

The second theorem we will illustrate says that a function $f(x)$ which is found to be continuous over a closed interval $[a, b]$ is guaranteed to have extreme values in that interval.

An extreme value of f, or *extremum*, is either a minimum or a maximum value of the function.

- A *minimum* value of f occurs at some $x = c$ if $f(c) \leq f(x)$ for all $x \neq c$ in the interval.
- A *maximum* value of f occurs at some $x = c$ if $f(c) \geq f(x)$ for all $x \neq c$ in the interval.

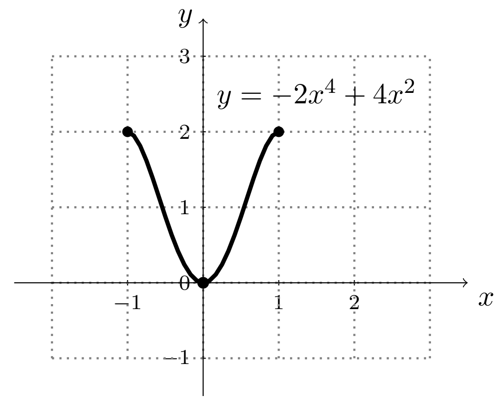
Theorem 5 (Extreme Value Theorem (EVT))**.** *If a function $f(x)$ is continuous over a closed interval $[a, b]$, then $f(x)$ is guaranteed to reach a maximum and a minimum on $[a, b]$.*

Note: In this section, we limit our illustration of extrema to graphical examples. More detailed and computational examples will follow once derivatives have been discussed.

EXAMPLE 4: Consider the function $f(x) = -2x^4 + 4x^2$ over $[-1, 1]$.

From the graph, it is clear that on the interval, f has

- The maximum value of 2, occurring at $x = \pm 1$; and
- The minimum value of 0, occurring at $x = 0$.



Remark 2: Similar to the IVT, the value $c \in [a, b]$ at which a minimum or a maximum occurs is not necessarily unique.

Here are more examples exhibiting the guaranteed existence of extrema of functions continuous over a closed interval.

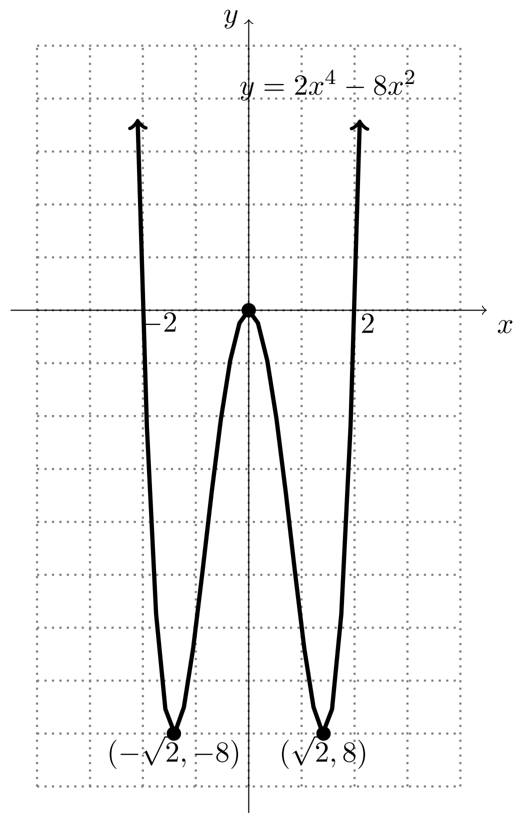
EXAMPLE 5: Consider Example 1. Observe that $f(x) = 2x - 5$ on $[1, 5]$ exhibits the extrema at the endpoints:

- The minimum occurs at $x = 1$, giving the minimum value $f(1) = -3$; and
- The maximum occurs at $x = 5$, giving the maximum value $f(5) = 5$.

EXAMPLE 6: Consider Example 2. $f(x) = x^2$ on $[-4, 2]$ exhibits an extremum at one endpoint and another at a point inside the interval (or, an *interior* point):

- The minimum occurs at $x = 0$, giving the minimum value $f(0) = 0$; and
- The maximum occurs at $x = -4$, giving the maximum value $f(-4) = 16$.

EXAMPLE 7: Consider $f(x) = 2x^4 - 8x^2$.



- On the interval $[-2, -\sqrt{2}]$, the extrema occur at the endpoints.
  - Endpoint $x = -2$ yields the maximum value $f(-2) = 0$.
  - Endpoint $x = -\sqrt{2}$ yields the minimum value $f(-\sqrt{2}) = -8$.
- On the interval $[-2, -1]$, one extremum occurs at an endpoint, another at an interior point.
  - Endpoint $x = -2$ yields the maximum value $f(-2) = 0$.
  - Interior point $x = -\sqrt{2}$ yields the minimum value $f(-\sqrt{2}) = -8$.
- On the interval $[-1.5, 1]$, the extrema occur at interior points.
  - Interior point $x = -\sqrt{2}$ yields the minimum value $f(-\sqrt{2}) = -8$.
  - Interior point $x = 0$ yields the maximum value $f(0) = 0$.
- On the interval $[-2, 2]$, the extrema occur at both the endpoints and several interior points.
  - Endpoints $x = \pm 2$ and interior point $x = 0$ yield the maximum value 0.
  - Interior points $x = \pm\sqrt{2}$ yield the minimum value -8.

Remark 3: Keep in mind that the IVT and the EVT are existence theorems ("*there is a value c ...*"), and their statements do not give a method for finding the values stated in their respective conclusions. It may be difficult or impossible to find these values algebraically especially if the function is complicated.

(C) **EXERCISES**

1. What value(s) of c, if any, will satisfy the IVT for the given function f and the given value m, on the given interval $[a, b]$. If there is (are) none, provide an explanation.

   a. $f(x) = x^2 - 1,\ m = 2,\ [-1, 2]$
   b. $f(x) = x^2 - 1,\ m = 2,\ [-1, 1]$
   c. $f(x) = x^3 + 2,\ m = 3,\ [0, 3]$
   d. $f(x) = \sin x,\ m = 1/2,\ [-\pi, \pi]$
   e. $f(x) = x^3 - 3x^2 + 3x - 1,\ m = -1,\ [-1, 2]$
   f. $f(x) = 4,\ m = 4,\ [-2, 2]$
   g. $f(x) = x,\ m = 4,\ [-2, 2]$
   h. $f(x) = x^2,\ m = 4,\ [-2, 2]$

2. Sketch the graph of each $f(x)$ in Item (a) to verify your answers.
3. Referring to your graphs in Item (b), where does each $f(x)$ attain its minimum and maximum values? Compute for the respective minimum and maximum values.
4. Determine whether the given function will have extrema (both a maximum and a minimum) on the interval indicated. If not, provide an explanation.

   a. $f(x) = x^2 - 1,\ (-1, 2)$
   b. $f(x) = |x|,\ [0, 1]$
   c. $f(x) = |x|,\ (0, 1)$
   d. $f(x) = \sin x,\ (-\pi, \pi)$
   e. $f(x) = \sin x,\ [-\pi/2, \pi/2]$
   f. $f(x) = x^3 - 3x^2 + 3x - 1,\ (-1, 1)$
   g. $f(x) = 1/x,\ [-2, 2]$
   h. $f(x) = [\![x]\!],\ [0, 1]$

⋆5. The next items will show that the hypothesis of the Intermediate Value Theorem – that f must be continuous on a closed and bounded interval – is indispensable.

   a. Find an example of a function f defined on $[0, 1]$ such that $f(0) \neq f(1)$ and there exists no $c \in [0, 1]$ such that
   $$f(c) = \frac{f(0) + f(1)}{2}.$$
   (*Hint*: the function must be discontinuous on $[0, 1]$.)
   Possible answer: Piecewise function defined by $f(x) = 1$ on $[0, 1)$ and $f(1) = 0$.

   b. Find an example of a function f defined on $[0, 1]$ but is only continuous on $(0, 1)$ and such that there exists no value of $c \in [0, 1]$ such that
   $$f(c) = \frac{f(0) + f(1)}{2}$$
   Possible answer: Piecewise function defined by $f(x) = 1$ on $[0, 1)$ and $f(1) = 0$.

$\star$6. The next items will show that the hypothesis of the Extreme Value Theorem – that f must be continuous on a closed and bounded interval – is indispensable.

a. Find an example of a function f defined on $[0, 1]$ such that f does not attain its absolute extrema on $[0, 1]$. (*Hint*: the function must be discontinuous on $[0, 1]$.)
Possible answer: Piecewise function defined by $f(x) = x$ on $(0, 1)$ and $f(0) = f(1) = \frac{1}{2}$.

b. Find an example of a function f that is continuous on $(0, 1)$ but does not attain its absolute extrema on $[0, 1]$
Possible answer: Piecewise function defined by $f(x) = x$ on $(0, 1)$ and $f(0) = f(1) = \frac{1}{2}$.

7. Determine whether the statement is *true* or *false*. If you claim that it is false, provide a counterexample.

a. If a function is continuous on a closed interval $[a, b]$, then it has a maximum and a minimum on that interval. Answer: True

b. If a function is discontinuous on a closed interval, then it has no extreme value on that interval. Answer: False, for example the piecewise function $f(x) = 0$ on $[0, 1/2]$ and $f(x) = 1$ on $(1/2, 1]$ achieve its extrema but it is discontinuous on $[0, 1]$.

c. If a function has a maximum and a minimum over a closed interval, then it is continuous on that interval. Answer: False, same counterexample as above

d. If a function has no extreme values on $[a, b]$, then it is discontinuous on that interval. Answer: True

e. If a function has either a maximum only or a minimum only over a closed interval, then it is discontinuous on that interval. Answer: True

8. Determine whether the given function will have extrema (both a maximum and a minimum) on the interval indicated. If not, provide an explanation.

a. $f(x) = |x + 1|,\ [-2, 3]$
b. $f(x) = -|x + 1| + 3,\ (-2, 2)$
c. $f(x) = [\![x]\!],\ [1, 2)$
d. $f(x) = [\![x]\!],\ [1, 2]$
e. $f(x) = \cos x, [0, 2\pi]$
f. $f(x) = \cos x, [0, 2\pi)$
g. $f(x) = x^4 - 2x^2 + 1,\ [-1, 1]$
h. $f(x) = x^4 - 2x^2 + 1,\ (-3/2, 3/2)$

$\star$9. Sketch a graph each of a random f over the interval $[-3, 3]$ showing, respectively,

a. f with more than 2 values c in the interval satisfying the IVT for $m = 1/2$.
b. f with only one value c in the interval satisfying the IVT for $m = -1$.
c. f with exactly three values c in the interval satisfying $m = 0$.
d. f with a unique maximum at $x = -3$ and a unique minimum at $x = 3$.
e. f with a unique minimum and a unique maximum at interior points of the interval.

f. f with two maxima, one at each endpoint, and a unique minimum at an interior point.

g. f with two maxima, one at each endpoint, and two minima occurring at interior points.

h. f with three maxima, one at each endpoint and another at an interior point, and a unique minimum at an interior point.

i. f with three zeros, one at each endpoint and another at an interior point, a positive maximum, and a negative minimum.

j. f with four maxima and a unique minimum, all occurring at interior points.

$\star$10. State whether the given situation is *possible* or *impossible*. When applicable, support your answer with a graph. Consider the interval to be $[-a, a]$, $a > 0$, for all items and that $c \in [-a, a]$. Suppose also that each function f is continuous over $[-a, a]$.

a. $f(-a) < 0$, $f(a) > 0$ and there is a c such that $f(c) = 0$.

b. $f(-a) < 0$, $f(a) < 0$ and there is a c such that $f(c) = 0$.

c. $f(-a) > 0$, $f(a) > 0$ and there is a c such that $f(c) = 0$.

d. f has exactly three values c such that $f(c) = 0$.

e. f has exactly three values c such that $f(c) = 0$, its minimum is negative, its maximum is positive.

f. f has exactly three values c such that $f(c) = 0$, its minimum is positive, its maximum is negative.

g. f has a unique positive maximum, a unique positive minimum, and a unique value c such that $f(c) = 0$.

h. f has a unique positive maximum, a unique negative minimum, and a unique value c such that $f(c) = 0$.

i. f has a unique positive maximum, a unique negative minimum, and two values c such that $f(c) = 0$.

j. f has a unique positive maximum, a unique positive minimum, and five values c such that $f(c) = 0$, two of which are $c = \pm a$.

k. f has two positive maxima, two negative minima, and no value c such that $f(c) = 0$.

l. f has two positive maxima, one negative minimum, and a unique value c such that $f(c) = 0$.

m. f has two positive maxima, one negative minimum found between the two maxima, and a unique value c such that $f(c) = 0$.

n. f has two maxima, two minima, and no value c such that $f(c) = 0$.

o. f has two maxima, two minima, and a unique value c such that $f(c) = 0$.

$\star$11. Determine whether the given function will have extrema (both a maximum and a minimum) on the interval indicated. If not, provide an explanation.

a. $f(x) = \sin x,\ (-\pi/2, \pi/2)$

b. $f(x) = \sin x,\ [-\pi/2, \pi/2)$

c. $f(x) = \dfrac{1}{x-1},\ [2, 4]$

d. $f(x) = \dfrac{1}{x-1},\ [-4, 4]$

e. $f(x) = \begin{cases} 2 - \sqrt{-x} & \text{if } x < 0, \\ 2 - \sqrt{x} & \text{if } x \geq 0, \end{cases}\ [-3, 3]$

f. $f(x) = \begin{cases} (x-2)^2 + 2 & \text{if } x < -1, \\ (x-2)^2 - 1 & \text{if } x \geq -1, \end{cases}\ [-3, 3]$

g. $f(x) = -x^4 + 2x^2 - 1,\ [-1, 1]$

h. $f(x) = -x^4 + 2x^2 - 1,\ \left(-\dfrac{3}{2}, \dfrac{3}{2}\right)$

TOPIC 4.3: Problems Involving Continuity

This is an OPTIONAL topic. It is intended for the enrichment of the students, to enhance their understanding of continuity and the properties it makes possible, such as stated in the Intermediate Value Theorem.

DEVELOPMENT OF THE LESSON

(A) INTRODUCTION

Continuity is a very powerful property for a function to possess. Before we even move on to its possibilities with respect to differentiation and integration, let us take a look at some types of problems which may be solved if one has knowledge of the continuity of the function(s) involved.

(B) LESSON PROPER

For every problem that will be presented, we will provide a solution that makes use of continuity and takes advantage of its consequences, such as the Intermediate Value Theorem (IVT).

APPROXIMATING ROOTS (Method of Bisection)

Finding the roots of polynomials is easy if they are special products and thus easy to factor. Sometimes, with a little added effort, roots can be found through synthetic division. However, for most polynomials, roots, can at best, just be approximated.

Since polynomials are continuous everywhere, the IVT is applicable and very useful in approximating roots which are otherwise difficult to find. In what follows, we will always choose a closed interval $[a, b]$ such that $f(a)$ and $f(b)$ differ in sign, meaning, $f(a) > 0$ and $f(b) < 0$, or $f(a) < 0$ and $f(b) > 0$.

In invoking the IVT, we take $m = 0$. This is clearly an intermediate value of $f(a)$ and $f(b)$ since $f(a)$ and $f(b)$ differ in sign. The conclusion of the IVT now guarantees the existence of $c \in [a, b]$ such that $f(c) = 0$. This is tantamount to looking for the roots of polynomial $f(x)$.

EXAMPLE 1: Consider $f(x) = x^3 - x + 1$. Its roots cannot be found using factoring and synthetic division. We apply the IVT.

- Choose any initial pair of numbers, say -3 and 3.
- Evaluate f at these values.

$$f(-3) = -23 < 0 \text{ and } f(3) = 25 > 0.$$

  Since $f(-3)$ and $f(3)$ differ in sign, a root must lie between -3 and 3.

- To approach the root, we trim the interval.
  - Try $[0,3]$. However, $f(0) = 1 > 0$ like $f(3)$ so no conclusion can be made about a root existing in $[0,3]$.
  - Try $[-3,0]$. In this case, $f(0)$ and $f(-3)$ differ in sign so we improve the search space for the root from $[-3,3]$ to $[-3,0]$.
- We trim further.
  - $f(-1) = 1 > 0$ so the root is in $[-3,-1]$.
  - $f(-2) = -5 < 0$ so the root is in $[-2,-1]$.
  - $f(-\frac{3}{2}) = -\frac{7}{8} < 0$ so the root is in $[-\frac{3}{2},-1]$.
  - $f(-\frac{5}{4}) = \frac{19}{64} > 0$ so the root is in $[-\frac{3}{2},-\frac{5}{4}]$.
- Further trimming and application of the IVT will yield the approximate root $x = -\frac{53}{40} = -1.325$. This gives $f(x) \approx -0.0012$.

The just-concluded procedure gave one root, a negative one. There are two more possible real roots.

FINDING INTERVALS FOR ROOTS

When finding an exact root of a polynomial, or even an approximate root, proves too tedious, some problem-solvers are content with finding a small interval containing that root

EXAMPLE 2: Consider again $f(x) = x^3 - x + 1$. If we just need an interval of length 1, we can already stop at $[-2,-1]$. If we need an interval of length $1/2$, we can already stop at $[-\frac{3}{2},-1]$. If we want an interval of length $1/4$, we stop at $[-\frac{3}{2},-\frac{5}{4}]$.

EXAMPLE 3: Consider $f(x) = x^3 - x^2 + 4$. Find three distinct intervals of length 1, or less, containing a root of $f(x)$.

When approximating, we may choose as sharp an estimate as we want. The same goes for an interval. While some problem-solvers will make do with an interval of length 1, some may want a finer interval, say, of length $1/4$. We should not forget that this type of search is possible because we are dealing with polynomials, and the continuity of polynomials everywhere allows us repeated use of the IVT.

SOME CONSEQUENCES OF THE IVT

Some interesting applications arise out of the logic used in the IVT.

EXAMPLE 4: We already know from our first lessons on polynomials that the degree of a polynomial is an indicator of the number of roots it has. Furthermore, did you know that a polynomial of odd degree has at least one real root?

Recall that a polynomial takes the form,

$$f(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$$

where $a_0, a_1, ..., a_n$ are real numbers and n is an odd integer.

Take for example $a_0 = 1$. So,

$$f(x) = x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n.$$

Imagine x taking bigger and bigger values, like ten thousand or a million. For such values, the first term will far outweigh the total of all the other terms. See, if x is positive, for big n the value of $f(x)$ will be positive. If x is negative, for big n the value of $f(x)$ will be negative.

We now invoke the IVT. Remember, n is odd.

- Let a be a large-enough negative number. Then, $f(a) < 0$.
- Let b be a large-enough positive number. Then, $f(b) > 0$.

By the IVT, there is a number $c \in (a, b)$ such that $f(c) = 0$. In other words, $f(x)$ does have a real root!

Teaching Tip

Ask the class why the claim may not hold for polynomials of even degree.

Answer: It is possible that the graphs of polynomials of even degree only stay above the x-axis, or only below the x-axis. For example, the graph of $f(x) = x^2 + 1$ stays only above the x-axis and therefore does not intersect x-axis, that is, $f(x)$ has no roots.

CHAPTER 1 EXAM

I. Complete the following tables of values to investigate $\lim_{x \to 1} (2x+1)$.

x	$f(x)$
0.5	
0.7	
0.95	
0.995	
0.9995	

x	$f(x)$
1.6	
1.35	
1.05	
1.005	
1.0005	

II. Using the tables of values above, determine the following:

1. $\lim_{x \to 1^-} (2x+1)$
2. $\lim_{x \to 1^+} (2x+1)$
3. $\lim_{x \to 1} (2x+1)$

III. Evaluate the following using Limit Theorems.

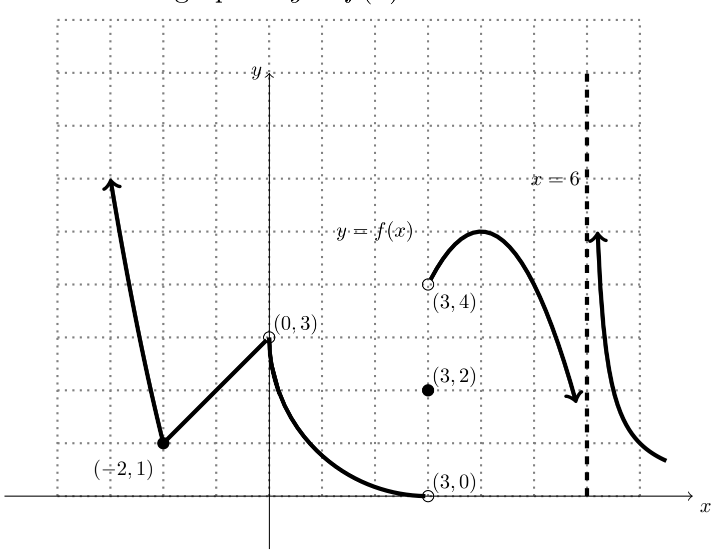
1. $\lim_{x \to 1} \dfrac{x^2 - 4x + 3}{x^2 - 1}$
2. $\lim_{t \to 0} \dfrac{3t - 2\sin t + (e^t - 1)}{t}$

IV. Let f be the function defined below.

$$f(x) = \begin{cases} \left| \dfrac{x^2 + 3x}{x+3} \right|, & \text{if} \quad x \leq 0, x \neq -3 \\ x + 1, & \text{if} \quad 0 < x < 1 \\ \sqrt{x}, & \text{if} \quad x \geq 1. \end{cases}$$

Discuss the continuity of f at $x = -3$, $x = 0$ and $x = 1$. If discontinuous, give the type of discontinuity.

V. Consider the graph of $y = f(x)$ below.



At the following x-coordinates, write whether (A) f is continuous, (B) f has a removable discontinuity, (C) f has an essential jump discontinuity, or (D) f has an essential infinite discontinuity.

1. $x = -2$
2. $x = 0$
3. $x = 3$
4. $x = 6$

Bibliography

[1] H. Anton, I. Bivens, S. Davis, *Calculus: Early Transcendentals*, John Wiley and Sons, 7th Edition, 2002.

[2] R. Barnett, M. Ziegler, K. Byleen *Calculus for Business, Economics, Life Sciences and Social Sciences*, Pearson Education (Asia) Pre Ltd, 9th Edition, 2003.

[3] L. Leithold, *College Algebra and Trigonometry*, Addison Wesley Longman Inc., 1989, reprinted-*Pearson Education Asia Pte. Ltd*, 2002.

[4] L. Leithold, *The Calculus 7*, Harpercollins College Div., 7th edition, December 1995.

[5] Math 53 Module Committee, *Math 53 Elementary Analysis I Course Module*, Institute of Mathematics, UP Diliman, 2012.

[6] J. Stewart, *The Calculus: Early Transcendentals*, Brooks/Cole, 6th Edition, 2008.

[7] S. Tan, *Applied Calculus for the Managerial, Life and Social Sciences*, Brooks/Cole, Cengage Learning, 9th Edition, 2014.

Biographical Notes

TEAM LEADER:

JOSE MARIA P. BALMACEDA

Dr. Jose Maria P. Balmaceda is a Professor of the Institute of Mathematics and Dean of the College of Science, University of the Philippines Diliman. He obtained the Ph.D. Mathematics degree from the University of Illinois at Urbana-Champaign and master's and undergraduate degrees in math from UP Diliman. He is a member of the National Academy of Science and Technology (NAST Philippines), Chair of the CHED Technical Committee for Mathematics, and President of the Mathematical Society of the Philippines. He is a recipient of various awards such as the NAST Outstanding Young Scientist Award, Science Prize (for Mathematics) from the Third World Academy of Sciences, Achievement Award from the National Research Council of the Philippines, and UP Diliman's *Gawad Chanselor* Awards for Most Outstanding Teacher and Most Outstanding Researcher.

WRITERS:

RICHARD S. LEMENCE

Dr. Richard S. Lemence is an Associate Professor and Deputy Director for Facilities and Resources, Institute of Mathematics, University of the Philippines Diliman. He obtained the doctorate degree in mathematics from Niigata University, Japan, specializing in differential geometry. He was a former faculty member of the Department of Mathematics of the Mindanao State University-Iligan Institute of Technology, where he also completed his baccalaureate and master's degrees in mathematics. Dr. Lemence was awarded by the National Academy of Science and Technology as Outstanding Young Scientist and designated by the University of the Philippines as U.P. Scientist I. He is the current National Director of the Philippine Mathematical Olympiad.

ORESTE M. ORTEGA JR.

Mr. Oreste M. Ortega Jr. is an Assistant Professor in the Mathematics Department of the Leyte Normal University, Tacloban City. He obtained his BS and MS in Mathematics degrees from the University of San Carlos in Cebu City. He is currently on dissertation writing for the Ph.D. Mathematics degree from the Ateneo de Manila University. Mr. Ortega has extensive experience in teaching calculus and other mathematics subjects, and is an active member of the Mathematical Society of the Philippines and other professional organizations. His research interests include graph theory and combinatorics.

CARLENE P. C. PILAR-ARCEO

Dr. Carlene P.C. Pilar-Arceo obtained her doctorate degree in Mathematics from the University of the Philippines Diliman (UPD), where she also finished her master's and baccalaureate degrees. She is currently a Professor at the Institute of Mathematics, U.P. Diliman and Program Development Associate of the UPD Office of the Vice Chancellor for Academic Affairs. She was former UPD College of Science Secretary and served as Assistant Secretary of the University. A sought-after resource person on the teaching and popularization of mathematics, Dr. Pilar-Arceo has also published scholarly articles on mathematical analysis and mathematical biology and two high school mathematics textbooks. She has received U.P. Diliman's *Gawad Chanselor* Award for Most Outstanding Teacher.

LOUIE JOHN D. VALLEJO

Dr. Louie John D. Vallejo is an Assistant Professor at the University of the Philippines Diliman. He holds a Ph.D. Mathematics degree from U.P. Diliman and master's and baccalaureate degrees (summa cum laude) from the same university. His doctoral dissertation research is on approximation theory and harmonic analysis. He has published his results in international journals and has presented his work in international and national conferences. Dr. Vallejo is currently the Deputy Team Leader of the Philippine delegation to the International Mathematical Olympiad and has been a trainer in the Mathematical Olympiad Selection Camp for several years now.

TECHNICAL EDITORS:

MARIAN P. ROQUE

Dr. Marian P. Roque is a Professor and Director of the Institute of Mathematics, University of the Philippines Diliman, where she also obtained her undergraduate, master's and doctorate degrees in mathematics. She occupies key positions in professional organizations such as membership in the Executive Committee of the *Centre International de Mathématiques Pures et Appliquées* (CIMPA), based in France, and the National Board of Directors of the Mathematical Society of the Philippines. She has received the Achievement Award from the National Research Council of the Philippines and UP Diliman's *Gawad Chanselor sa Natatanging Guro*. She holds a designation as UP Scientist I and is member of the UP Diliman General Education Committee. Dr. Roque co-authored the graduate textbook Introduction to Classical and Variational Partial Differential Equations published by the UP Press.

JOSE ERNIE C. LOPE

Dr. Jose Ernie C. Lope is a Professor and former Chair of the then Department of Mathematics, University of the Philippines Diliman. He obtained master's and doctorate degrees in mathematics from Sophia University, Japan. A Bachelor of Science Mathematics summa cum laude graduate of UP Diliman, Dr. Lope has received awards for his research and teaching such as the UP Scientist I designation, Outstanding Young Scientist Awards from the NAST and the Third World Academy of Science and the *Gawad Chanselor sa Natatanging Guro* from UP Diliman. A former member of the Philippine Team to the International Mathematical Olympiad (IMO), he later became Team Leader of three Philippine delegations to the IMO that set Philippine records for medals and citations garnered by the team members.

LANGUAGE AND COPY EDITOR:

ERICK B. LIRIOS

Mr. Erick B. Lirios currently teaches journalism and photojournalism at the Manila Times College, and works as a columnist for The Manila Bulletin. He edited academic books for McGraw-Hill Education, the Department of Interior and Local Government and the Ateneo School of Government. He worked as the Communications Executive for Singapore Telecommunications, as the Corporate Communications Executive for the Spastic Children's Association of Singapore, and the Editor-in-Chief of Telecable Magazine. Besides doing proofreading, he also works as a photographer - the latest being with the World Health Organization.